GENERAL PROBLEMS

IN THE

LINEAR PERSPECTIVE

OF

FORM, SHADOW, AND REFLECTION;

OR THE

SCENOGRAPHIC PROJECTIONS OF DESCRIPTIVE GEOMETRY.

BY

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PREFACE.

Perspective has been, in some respects, an unfortunate member of the company of Sciences. For, as perspectives are not made, except by the eye, from objects themselves, but from adequate substitutes for them, and as projections alone, by the methods of the modern science of Descriptive Geometry, perfectly furnish such substitutes; every system of perspective, that is not expressly based on the theory of projections, is more or less inelegant, tedious, blind, and cumbersome.

Any one, then, who compares Pl. XVII., Fig. 111, for example, and its companion Fig. 112, or "tender," to supply meaning to it, with Pl. XVII., Fig. 113, will understand how the mind, trained in the modern methods of Descriptive Geometry, loves the fixed "horizontal plane" to stand upon, and the "vertical plane" to lean upon—in simple words, two fixed planes of reference—rather than a single such plane, into which, as in Pl. XVII., Fig. 111, everything must be revolved and treated in its revolved position.

But now that the difficulties here noted, and which have done much to render perspective a hitherto unpopular study, have been removed, it is to be hoped that it will soon come to be welcomed; both as a *valuable disciplinary instrument* for the mind, in common with all the other components of Descriptive Geometry, as well as on account of the utility and beauty of its technical applications.

Agreeably to the views just expressed, no attempt has been made, in this work, to exhibit the old, unsystematic, and perplexing methods, fully; however ingenious, sometimes, in themselves, and interesting historically; though a few suggestions have been derived from them, and a few examples given, of constructions by them.

Having thus indicated that this work is essentially modern in *spirit* and *method*, a word follows as to its range and scope.

This work is not meant to be merely a new version of existing works; nor has novelty been sought, either in its matter or form, at the expense of rejecting such *standard methods and problems as must enter* into every fairly complete treatise. Yet, as far as seemed desirable, it

has been intended to be, either in substance or treatment, an independent contribution to existing sources of information about the higher principles and practice of perspective construction.

In this regard, four principal particulars have been kept in view. *First*, to make a full exhibition, as in Chapter II., of the numerous general methods of perspective, serving as an introduction to many writers on the subject. *Second*, to present a good and fully treated collection of standard general and special problems, on perspective, including a view of the various practical expedients for abbreviating and simplifying perspective constructions, in order to qualify the student to make all the constructions that would be likely to occur in practice. *Third*, to take some one or two higher problems, as that of the sphere, and the torus, and to treat them so exhaustively, that, when fully mastered, they might contribute somewhat to the student's power to investigate for himself. *Fourth*, to afford an introduction to some of the special and curious applications of perspective, as the perspectives of reflections, of the shadows cast by candle-light; and distorted perspectives.

The writer's aim has thus been, to leave nothing to be desired in respect to fulness of elementary illustrations of all known methods of procedure; or sufficiency of varied examples for practice.

The full and separate treatment of the subject of Chap. IV., Section iv., was suggested by a careful reading of the purely practical, but richly illustrated, treatise by J. ADHEMAR. The collection of beautiful examples, given in his atlas, being wrought by the simple method of scales, serve to connect perspective, as a disciplinary study and body of theoretical science, with its use as a thoroughly available practical art, both for artistic and industrial purposes.

The present volume, being principally intended for the instruction of the scientific student, its diagrams are made, as usual in text-books, with all the apparent distortions due to an unnaturally small distance of the eye from the perspective plane.

I have been careful, as in my "Shades and Shadows," to favor the convenient learning of the Problems by dividing them up into numbered topics; the separate topics among the groups of general principles are also separated by blanks in the text.

By the frequent adoption of *skeleton figures* of objects, instead of their usual solid forms, many uninstructive repetitions of the same step in construction have been avoided, and the cost of the plates reduced, without diminishing their real value.

The genius of the methods of Pl. XVII., Figs. 110, 111, p. 58, consists in rendering the eye of the supposed spectator at *e*, indepen-

dent of the student of the diagram. It thus illustrates the tendency of the English mind to render everything "objective," or sensibly separate from the person of the thinker.

The modern, or French, methods, are more subjective, as in all the figures, generally, of this work, where the student is supposed to make EE′, the point of sight, the position of his own eye.

By a very kindly application of the principle of "division of labor," the comparatively heavy expense, to the author, of a work like this, has been generously shared, in the present instance, by a company of liberal friends; whose favor, equally to himself and to the Institute, in this their action, he hereby acknowledges with sincere thanks.

R. P. I., June, 1868.

LINEAR PERSPECTIVE.

CHAPTER I.

General Principles, and Introductory Problems.

1. PERSPECTIVE, defined as an *imitative art*, is the art of representing objects on some surface, usually a plane, so that the representation shall make, as nearly as possible, the same impression on the eyes, placed in a given position, that the objects themselves would, when seen from the same position. These representations are called *perspectives;* the plane on which they are made, the *perspective plane.*

2. Familiarity with the stereoscope shows that the impression of rotundity, and of relative distances from the observer, is due to the blending, in the internal perception, of the two slightly different images of the same object, formed in each eye.

The partial lack of this gratifying impression of solidity, in viewing even a perfectly drawn single flat picture, may be supplied in part by the imagination, and in great part by viewing the picture through a tube, which, by excluding the sight of all other objects, aids in abandonment of mind to the fullest impression which the picture can make.

The necessity of a certain kind of development of the imagination, as just noted, is confirmed by the fact that some persons can see little or nothing more in pictures than an unmeaning assemblage of various patches of different colors or shades.

3. The position of the eyes is called the *point of sight.* For, in constructing perspectives, the eyes are supposed to be reduced to a single seeing point; *first*, because the observer holds a single position, when looking at a given view, as a whole; and

second, because, though each eye forms its own image, but a single impression is received by the mind.

4. The visible boundary of any object, as seen from a given point of sight, is called its *apparent contour*.

5. Objects become visible by means of such rays of light, returned from them, as enter the eye. These rays, like all others, proceed in straight lines, and are called *visual rays*.

6. The intersection of the visual ray from any point with the perspective plane, is the perspective of that point; for this intersection, called the *trace* of the ray, exactly covers and conceals from view the original point, and therefore represents it, according to (1).

7. The apparent contour of a body is hence seen to be its curve of contact with a circumscribing tangent cone, whose vertex is the point of sight. This cone is called *the visual cone*, since each element of its surface is evidently a visual ray.

8. Linear Perspective, which is a branch of Descriptive Geometry, may therefore now be defined as the general problem of finding the *intersection of a given plane with a given cone;* the given plane being the *perspective plane*, the vertex of the cone being the *point of sight*, and its base the *apparent contour* of the given object. Moreover, the vertical plane of projection may always be taken as the perspective plane.

9. The *practical* conclusion from the preceding principle is, that we must know by measurement the actual sizes and distances of real objects, before we can make true perspectives of them, as seen from a given point of sight.

10. But in perspective, as in other parts of Descriptive Geometry, objects are not given actually, but by their projections. Having, then, the relative position of the eye and the given object truly shown in projection, together with the dimensions and distance of the latter, drawn to a suitable scale from actual measurements, we can find the true perspective of that object, as seen from the given point.

11. The horizontal projection of the point of sight is frequently called the *station point*. Its vertical projection, the *centre of the picture*.

The eye, or the axis of vision, is generally supposed to be directed toward the centre of the picture. The visual ray e E′,

Pl. XVI., Fig. 108, from the centre of the picture, may be called the *central visual ray*.

12. We can find the perspective of an object under the conditions just stated (in 10), for, by the problem (IV. Des. Geom.), to find where a given line pierces the planes of projection, we can find where visual rays, from the distinguishing points of the object, pierce the perspective plane, and these intersections will be the determining points of the perspective figure.

13. The perimeter of the perspective is the perspective of the apparent contour (7) of the given body; hence the latter must be known before the former can be found.

The apparent contour of any *plane sided body*—polyedron—can generally be found by *inspection*. It consists of those edges which appear to divide the object from surrounding space. That of a *developable surface*, consists of the elements of contact of tangent visual planes; and that of a *warped*, or a *double curved surface*, of the curve of contact of a tangent circumscribed visual cone. This curve of contact (D. G., Prob. XC.) is practically found by constructing a series of visual *planes*, tangent to the given surface, and joining their points of contact.

14. Having thus found the linear perspective of an object, if we shade and color it like the original, as nearly as art will allow, it will, at each of its points, present to the eye the same appearance, as to form and color, as the original; for each point of the perspective, being on the same visual ray with the corresponding point on the given body, will fitly replace it, and having also the same color, will fully represent it.

The department of perspective which embraces shading and coloring, is *Aërial Perspective*. It is founded on careful observation, and belongs more to the artistic than to the scientific draftsman, and will, therefore, be mostly neglected in this volume.

15. Let us then next consider what the visual cone, and the perspective figure, will reduce to, in case of various forms of the original object. When the object is a *point*, the visual cone becomes a single visual ray, and the perspective is a point, as before stated (6). In case of a *limited straight line*, the visual cone reduces to a plane triangle, whose base is the given line, and the perspective is a straight line. The object being *a straight line of indefinite extent*, the visual cone will be an

indefinite plane, and the perspective, the indefinite straight line in which this plane intersects the perspective plane. When the object is *a plane figure*, or *a plane sided solid*, the visual cone is strictly a pyramid, and the perspective figure a plane figure. If, however, the given plane figure, whether a polygon or curved, includes the point of sight in its own plane, the visual cone becomes a plane, and the perspective, a straight line. Finally, when the given object is *a curve*, or *a curved surface*, the visual cone is, in general, a cone in the ordinary sense of having a curved base; though, if the given surface be developable, its apparent contour will consist of rectilinear elements, and the visual cone of two tangent planes, as to a cylinder.

16. The perspective of any magnitude being thus the intersection of the visual surface passed through it, with the perspective plane, *the perspective of a straight line is, really, the trace, on the perspective plane, of the visual plane containing it.*

17. If, then, such a line be *parallel* to the perspective plane, *its perspective will be parallel to the line itself;* and if any number of such parallels be also parallel to each other, *their perspectives will be parallel to each other, and to the lines themselves.*

18. If a line be *perpendicular* to the perspective plane, its visual plane will be perpendicular to that plane, also. Hence this visual plane will be the projecting plane of the visual rays from all points of the line. The perspective of the line will therefore coincide with the vertical projection of the visual ray from any of its points; *and will consequently pass through the vertical projection of the point of sight.*

19. The visual planes, containing any number of such perpendiculars, all being perpendicular to the perspective plane, and all containing the point of sight, will all intersect each other in the line which projects that point upon the perspective plane. Hence their vertical traces, which are the perspectives of the lines, will all contain the vertical projection of the point of sight. That is, *the perspectives of all perpendiculars to the perspective plane, pass through the vertical projection of the point of sight.*

20. It must here be remembered, that the *apparent size* of any object depends upon the angle, at the eye, subtended by it. This angle is called the *visual angle.* Therefore—

First, all the different sections of the visual cone, made by the perspective plane, will subtend the same visual angle at the eye, placed at the cone's vertex. Hence, though of very different *actual* sizes and forms, according to their distances from the eye, and their positions relative to the cone's axis, they will all be of the same *apparent* size and form, when seen from the point of sight. The perspectives, cut from a visual cone by parallel planes will be *similar*, but unequal. But if these planes be not parallel, the perspectives will vary both in *form* and *size*.

Second. Objects of very *different actual size* may have the *same apparent size*, if placed so as to subtend the same visual angle. Conversely, objects of the *same actual size* may have *different apparent sizes* if so placed as to subtend different visual angles.

21. The particulars which affect the visual angle, subtended by a given body, and consequently, its apparent size, are two: its *distance* from the eye, and the *direction*, as perpendicular or oblique to it, in which it is viewed. The effect upon the apparent size of an object, which results from viewing it obliquely, is called *foreshortening*.

22. There now follows from the second of the above principles (20) the additional one, that the sense of sight alone, judging from the *apparent* sizes alone, of objects, teaches nothing as to their *actual* sizes, or distances from us. Knowledge of the latter facts is obtained mostly through the *sense of touch*, *exercised* in actually measuring sizes and distances, or *trusted* in much of our knowledge, however acquired, of relative sizes and distances. Knowledge of actual sizes and distances also arises from the comparative brightness or dulness of similar objects, seen in the same light, and by the manner in which they conceal one another, or shade one another, and from our knowledge of the law of gravity, as when we know that the apparent and real contact of a chair with the floor coincide, since the chair cannot support itself in mid-air.

23. In dismissing the subject of the visual angle (20) and its relations, it must be noted that this angle may be regarded as *solid*, *diedral*, or *plane*. The vertex of the visual cone is a solid visual angle; the visual angle between two tangent visual planes to a developable surface is diedral; and any plane through the vertex of the visual cone gives a plane visual angle, included by the elements of the cone contained in that plane.

24. The visual angle very commonly means the one formed by lines cut by a *horizontal* plane, through the vertex of the visual cone, that, is through the eye, from the vertical visual planes containing the extreme right and left hand points of the object drawn.

25. In the following problems, and generally in perspective constructions, *the perspective plane is placed between the eye and the object*, that is, between the vertex and the base of the visual cone. The perspective being thus made smaller than the object, any instrumental errors in the projections of the latter will be reduced in the perspective; whereas, if the perspective plane were beyond the object, these errors would be magnified. Neglecting these errors, however, the perspective would be equally true in either case.

26. All the preceding articles can be readily understood without pictorial, or oblique projections, by any one familiar with descriptive geometry, but for general convenience, the leading elementary principles will be presented again, in connection with the following introductory problems, and illustrated by a part of the pictorial model seen in Pl. XVI., Fig. 108.

Let RR represent the horizontal plane; GHL, the (vertical) perspective plane; *e*, the eye, and M, any point in the horizontal plane.

Then M*e* is the visual ray from M; and M″, its intersection with the perspective plane, is the perspective of M (6).

E and E′ are the horizontal and vertical projections of the point of sight (3). GL is the ground line, and MM′ is drawn from M, perpendicular to the perspective plane; and is also the horizontal trace of a visual plane through M,M′. This plane contains *e*,E′, hence its vertical trace, M′E′, which is the perspective of MM′, passes through E′, according to (19).

The construction of M″ is also obvious, being projected from Q, where the horizontal proiection, ME, of the visual ray meets the ground line, into M′E′, the vertical projection of the same ray.

Example 1.—*Construct a pictorial figure, similar to Pl. XVI., Fig.* 108, *in which M shall be in space.*

Ex. 2.—*Also one to illustrate that the perspectives of any lines parallel to the perspective plane, and to each other, will be parallel.*

Ex. 3.—*Also one to illustrate that the perspectives of per-*

pendiculars to the perspective plane pass through the centre of the picture.

Problem I.

To find the perspective of a pyramid, and its pedestal, by visual rays.

Remark.—In all the problems of this book, a point, etc., is briefly named by naming its projections. Thus: the point aa' means the point whose horizontal projection is a, and whose vertical projection is a'; the line $ab-a'b'$ has a like meaning; and the words, "the point aa' describes the arc $ac-a'c'$," mean that the point whose projections are a and a', describes the arc whose horizontal projection is ac, and whose vertical projection is $a'c'$.

Let $ahd-a'b'c'd'$, Pl. I., Fig. 1, be the projections of the pedestal; the square on fg, and its diagonals, the horizontal projection of the pyramid; and $v'f'g'$, its vertical projection. Also let kL be the ground line, let the vertical plane, thus taken a little in front of the axis of the pyramid, be also the perspective plane, and let E,E′ (E is below Fig. 5) be the point of sight. The apparent contour of the given body is obvious on inspection.

Beginning, now, with the upper left hand corner a,a' of the pedestal, aE$-a'$E′ is the visual ray from that point. By (Prob. IV., D. G.)—to find the intersection of a given line with either plane of projection—this ray pierces the vertical, or perspective plane, at A, which is therefore the perspective of a,a'. (This construction, being of constant application throughout all parts of descriptive geometry, is here explained in detail, at the outset, once for all, as follows:) The intersection, k, of the horizontal projection of this ray with the ground line, is, since it is in the ground line, the horizontal projection of that point of the ray which is in the vertical plane, which is the point sought. This latter point, being also a point of the ray, its vertical projection, which is also the point itself, is the intersection of the projecting line through k with the vertical projection, a'E′, of the ray. This projecting line, which is perpendicular to the ground line, will meet the vertical projection of the ray, in any given case, above or below the ground line, according

to the relative position of the projections of the given ray and the ground line. Thus, the ray $aE-b'E$, from the lower left corner, a,b', of the pedestal, meets the vertical or perspective plane below the ground line, at B, the perspective, therefore, of a,b'.

The perspective of every point of the object might be likewise found; but, the front edges of the pedestal being parallel to the perspective plane, their perspectives are (17) parallel to those edges; and here observe, that when we know the *direction* of a perspective line, we can find its limits, or any point on it, by using either projection, alone, of the visual ray from that point. Thus, draw AC, indefinitely, parallel to $a'c'$, then to find C, produce $E'c'$ till it meets AC, at C, the perspective of d,c'. Or, produce Ed (not shown) to the ground line, and thence draw the perpendicular which will meet AC at C.

According to (19) AH is the perspective of the edge $ah-a'$.

The student should complete the construction here begun. As the perspective plane is taken through the body of the pyramid, the perspective of those parts which are in front of that plane, will by (25) be larger than the parts themselves. Thus AC is longer than $ad-a'c'$.

27. Without proceeding any further with the method of visual rays, in the manner just illustrated, its inconvenience can clearly be seen: the two projections, the perspective, and the lines of construction, being all confounded together; unless the point of sight is placed so far to one side as to bring the perspective, and much of the construction, clear of the projections. But, as the perspective plane is supposed to be viewed perpendicularly, and toward the vertical projection of the point of sight, the position of the eye just named is quite objectionable, since it corresponds to a very oblique vision of the given object, and hence to a very indistinct view of it.

28. The next thing to be sought is, therefore, a remedy for the confusion to lines just observed. This is found in new arrangements of the planes of projection, of which the following are the principal ones:

First. The perspective plane can be *translated*, that is moved parallel to itself, *forward*, or *toward* the eye, till it can be revolved back, as usual, into the horizontal plane, without bringing down the perspective figure upon the projections.

Second. The perspective plane may be similarly translated *backward*, and then revolved.

Third. Without translating the perspective plane, the horizontal plane can be revolved 180°, bringing the plan of the given original object in front of the ground line.

29. All of these methods leave two of the three figures (the two projections, and the perspective) in the same part of the paper; still, they suffice very well in finding the perspectives of simple objects. But as they only partially remedy the difficulty encountered, we have, further, the following arrangement:

Fourth. Make the perspective plane separate from both planes of projection, and, for the greatest convenience, perpendicular to them. This method is most advantageously applied in finding the perspectives of quite complex objects, or of double curved surfaces, which require considerable preliminary construction to determine their apparent contour (13).

30. But a further examination of Pl. I., Fig. 1, reveals another difficulty.

The most natural manner of viewing an object, is to look directly, and horizontally, toward a central point in it, the visual ray from which should, if possible, be the *central visual ray* (11). Then, as the perspective plane is supposed to be viewed perpendicularly, its best position is perpendicular to this central visual ray, or to its horizontal projection. Thus, in Pl. I., Fig. 1, kL would better be perpendicular, or nearly so, to Ev, or E,E′ should be placed so as to make Ev perpendicular to the present position of kL.

But, in either case, the construction by visual rays, as shown at A, would give rise to very acute, and therefore poorly defined intersections, for most of the points sought. Hence *other methods of construction*, as well as other dispositions of the perspective and other planes, are necessary. These new methods we will now proceed to unfold.

THEOREM I.

The perspective vanishing point of a system of parallel lines, is the intersection of a visual ray, parallel to them, with the perspective plane.

31. If visual planes, as AB—E, and CD—E, Pl. I., Fig. 2, be passed through each one of a system of parallel lines, as

AB, CD, etc., they will intersect each other in a visual ray, ER, parallel to the given lines. The traces, BV, EV, etc., of these visual planes, on the perspective plane, are the perspectives (16) of the lines, and these traces will all meet at V, where the parallel ray, ER, common to all the planes, meets the perspective plane. This point will be the perspective of the point R, at an infinite distance from E, where this parallel visual ray meets the given parallels of the point R. Now R, being at an infinite distance, is the *real vanishing point, in space*, of AB, etc., where they disappear from sight; but V, its perspective, only being accessible for use, is commonly called the vanishing point.

32. Hence, to find the vanishing point of any line, or system of parallels, we have the following rule: *Draw a visual ray parallel to these lines, and find its intersection with the perspective plane; which will be the point required.*

33. The following results obviously arise from the theorem just explained:

1st. All *perpendiculars* to the perspective plane have the *centre of the picture* (11) for their vanishing point, as otherwise proved in (19).

2d. All *horizontal lines* have their vanishing points in a horizontal line through the centre of the picture. This line is called the *horizon.*

3d. All *parallels* to the perspective plane will have no vanishing point. Hence their perspectives will be parallel to the lines themselves (17).

34. In Pl. XVI., Fig. 108, now, for example, MN, and MK, are horizontal lines, and HE′, parallel to GL, is the horizon (33). Hence we see that eV, a visual ray parallel to MN, and eV′, parallel to MK, pierce the perspective plane at V and V′, the vanishing points, in the horizon, of the given lines (33). Likewise, it is clear that E′, the centre of the picture (11) is the vanishing point of MM′, and of all perpendiculars.

EXAMPLE.—*Construct a figure, similar to the last, but in which the given lines shall be parallel in any direction in space.*

THEOREM II.

If a given point be at the intersection of two lines, its perspective will be at the intersection of their perspectives.

35. For these perspectives are the traces, on the perspective plane, of the visual planes through those lines. But these planes intersect each other in a visual ray, which will contain the given point, and the intersection of their traces. Hence the latter point, being thus shown to be the intersection of a visual ray with the perspective plane, it will be the perspective of the given point, through which that ray passes.

THEOREM III.

The perspective of a straight line joins the trace of the line with its vanishing point.

36. The vanishing point of such a line has already been shown to be one point of its perspective, and as two points determine the perspective, it is only necessary to show that the *trace* of the line, that is, its intersection with the perspective plane (6), is also a point of its perspective.

37. Now *the intersection of any line with the perspective plane is a point of its perspective*, since the line intersects the perspective plane in the trace of the visual plane through it; and this trace is the indefinite perspective of the line.

38. Otherwise: The point in which a line pierces the perspective plane, briefly called its *trace* on that plane, is one point of its perspective, since the visual ray *from* that point pierces the perspective plane immediately *at* that point. In short, the point supposed is its own perspective.

39. In illustration of this theorem, and of (35), see Pl. XVI., Fig. 108.

First, KV′ is the perspective of MK, since K is its trace, or intersection with the perspective plane, and V′ is its vanishing point. Likewise, NV is the perspective of MN (35). Hence, *second*, M″, the intersection of the perspectives, KV′ and NV, is the perspective of M, the given point through which MK and MN were passed.

The two theorems, just demonstrated, should be made thoroughly familiar, as they are of constant use in finding the perspectives of the auxiliary lines used in constructing the perspectives of points.

EXAMPLE.—*Let the above figure be constructed with M in any position in space.*

40. Principles, analogous to those concerning vanishing points of lines, are true in respect to *planes*. Thus, visual rays, parallel to any two lines in a given plane, will, by (31), pierce the perspective plane in the vanishing points of those lines. But these visual rays determine a visual plane, parallel to the given plane, and which will therefore contain the visual rays parallel to *all* lines in the latter plane. Hence the vanishing points of all lines of the given plane, will be in the trace, on the perspective plane, of the parallel visual plane. Hence this trace is called the *vanishing line* of the given plane.

41. Otherwise: Parallel planes actually vanish together, in a line of intersection at an infinite distance. That one of these planes which passes through the eye is the visual plane, whose trace on the perspective plane is the perspective of that infinitely distant vanishing line in space. This perspective is, itself, for convenience, called the vanishing line of the given planes.

42. Briefly summing up the two last articles, we have these principles:

1°. All parallel planes have the same vanishing line.

2°. This vanishing line is the trace, on the perspective plane, of a parallel visual plane.

3°. This trace is determined by the points in which two visual rays parallel to *any* lines in the given plane, or planes, pierce the perspective plane. Hence,

4°. The vanishing line of any parallel planes, contains the vanishing points of all lines in those planes. Or, all lines in any plane, *or parallel to it*, have their vanishing points in the vanishing line of that plane.

5°. *Any* line, through the centre of the picture, is the vanishing line of all planes parallel to it, and perpendicular to the perspective plane.

6°. Therefore, *the horizontal line*, is only that case of vanishing line just mentioned, which corresponds to *horizontal* planes, and has no general properties different from those of the vanishing lines of *any* planes, perpendicular to the perspective plane.

43. For purposes of construction, we may now express the preceding principles in the following—

THEOREM IV.

The vanishing line of any plane is a parallel to the vertical trace of that plane, through the vanishing point of its horizontal trace.

For, *first*, parallel planes have parallel traces, and the required vanishing line is the vertical trace of a *visual* plane parallel to the given plane. Hence this line is parallel to the vertical trace of the given plane, as stated.

But, *second*, this vanishing line also contains the vanishing points of all lines in the given plane, and the horizontal trace of that plane is such a line, and one whose vanishing point is most simply found, it being in the horizon. Hence the Theorem is proved.

PROBLEM II.

To find the vanishing line of given parallel planes, and the perspective of a line in one of the planes.

44. Let PQP′, Pl. VII., Fig. 59, and RSR′, be the given planes, and EE′, the point of sight. The required vanishing line is the trace parallel to P′Q, on the perspective plane, of a visual plane parallel to the given planes. Draw the ray E*k*—E′*k*′, parallel to PQ and RS, the horizontal traces of the given planes. It pierces the perspective plane at *k*′, through which *k*′*v*′, parallel to QP′, is the required vanishing line.

EXAMPLE.—*Construct k′v′ by the general method of* (42—3°).

45. All lines in a plane, or in parallel planes, vanish in the vanishing line of that plane, or those planes (42—4°). Thus let R*n*—*m*′*n*′ be a line in the plane RSR′. Its vanishing point, *v*′, being on *k*′*v*′, is therefore found by drawing either projection, only, of the visual ray parallel to it. Thus *v*′ is the intersection of E′*v*′, parallel to *m*′*n*′, with *k*′*v*′, or of the vertical *vv*′ with *k*′*v*′, observing that E*v* is parallel to R*n*. But *n*′, the intersection of the given line with the perspective plane, is a point of

its perspective. Hence *n'v'* is the indefinite perspective of R*n*—*m'n'*.

Again, parallel lines have the same vanishing point, hence—*p'v'* is the perspective of *op*—*o'p'*, a parallel to R*n*—*m'n'*.

46. We will here introduce some examples, of the more elegant kind, of the earlier methods of construction, which are distinguished by employing, generally, but one fixed plane of projection, viz., the perspective plane, into which various auxiliary planes are revolved.

Problem III.

Having given the centre and distance of the picture; and the inclination of a given plane to the perspective plane, together with the direction of its trace on that plane; to find the vanishing line of the given plane.

First. When the given plane is parallel to the ground line, Fig. 1.

Let EH be the horizon, E the centre of the picture, EE″ its distance from the eye, and θ the complement of the inclination of the plane to the perspective plane. E″ is here the point of sight revolved horizontally about its vertical projection, E, as a centre, and into the horizon. Then E″M is simply the revolved position of the visual ray, parallel to the given plane, and making the same angle with the perspective plane, that the given plane does. Hence M is a point of the required vanishing line, which, being parallel to the horizon, is so drawn, as at MV.

Second. When the trace of the given plane, on the perspective plane, has any direction, as TT, Fig. 2.

Simply consider that EE″, parallel to the given direction, TT, has the same relation to the required vanishing line, that EE″ has in Fig. 1. That is, let KEM be the vertical trace of a plane, perpendicular both to the perspective and given planes, and revolve this plane about that trace and into the perspective (vertical) plane; and E″ will be the point of sight, revolved into the latter plane; and EE″ will show its true distance from the perspective plane. Then E″M being, as before, the revolved visual ray, parallel to the given plane, and showing at EME″ the angle included by the perspective and given planes, M is one point of the required vanishing line MV, which is parallel to EE″.

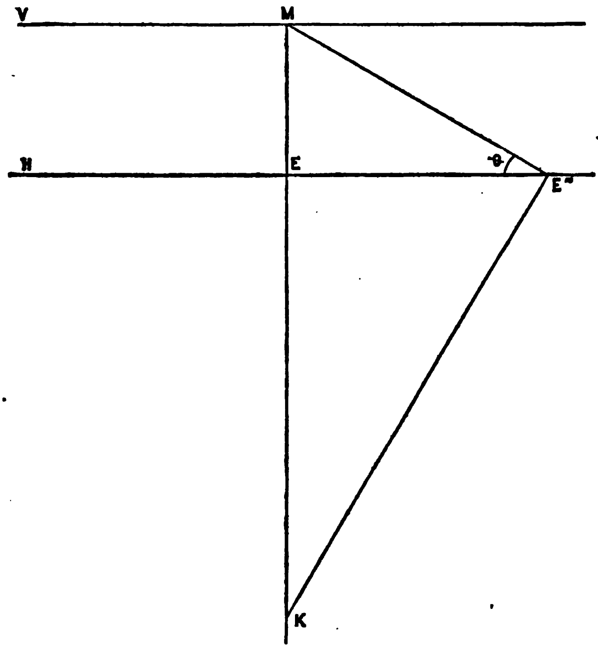


FIG. 1.

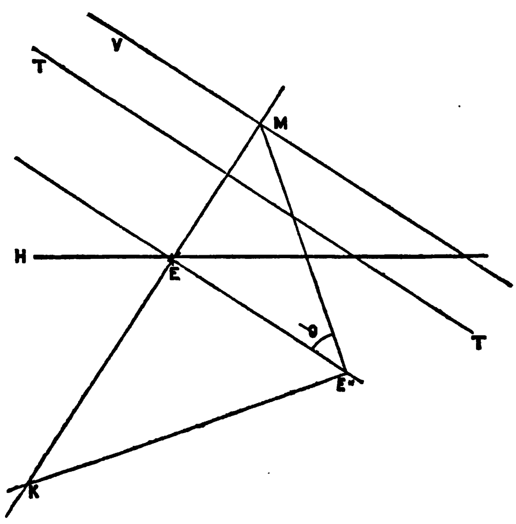


FIG. 2.

Problem IV.

Having given the centre, and distance, of the picture, to find the vanishing point of all perpendiculars to a given plane; or the vanishing line of all planes perpendicular to a line whose vanishing point is known.

First. In both the preceding figures, E″K, perpendicular to E″M, is the visual ray, perpendicular to the plane whose vanishing line is MV, and whose inclination to the perspective plane is E″ME. And this ray evidently pierces the perspective plane as at K, in the trace of the perpendicular plane, KEM, containing it. Hence K is the vanishing point of the perpendiculars required.

Second. K being given, and E″ found, as above, draw KE and produce it, and also E″M perpendicular to KE″, and M will be a point of the required vanishing line MV which will be perpendicular to KE.

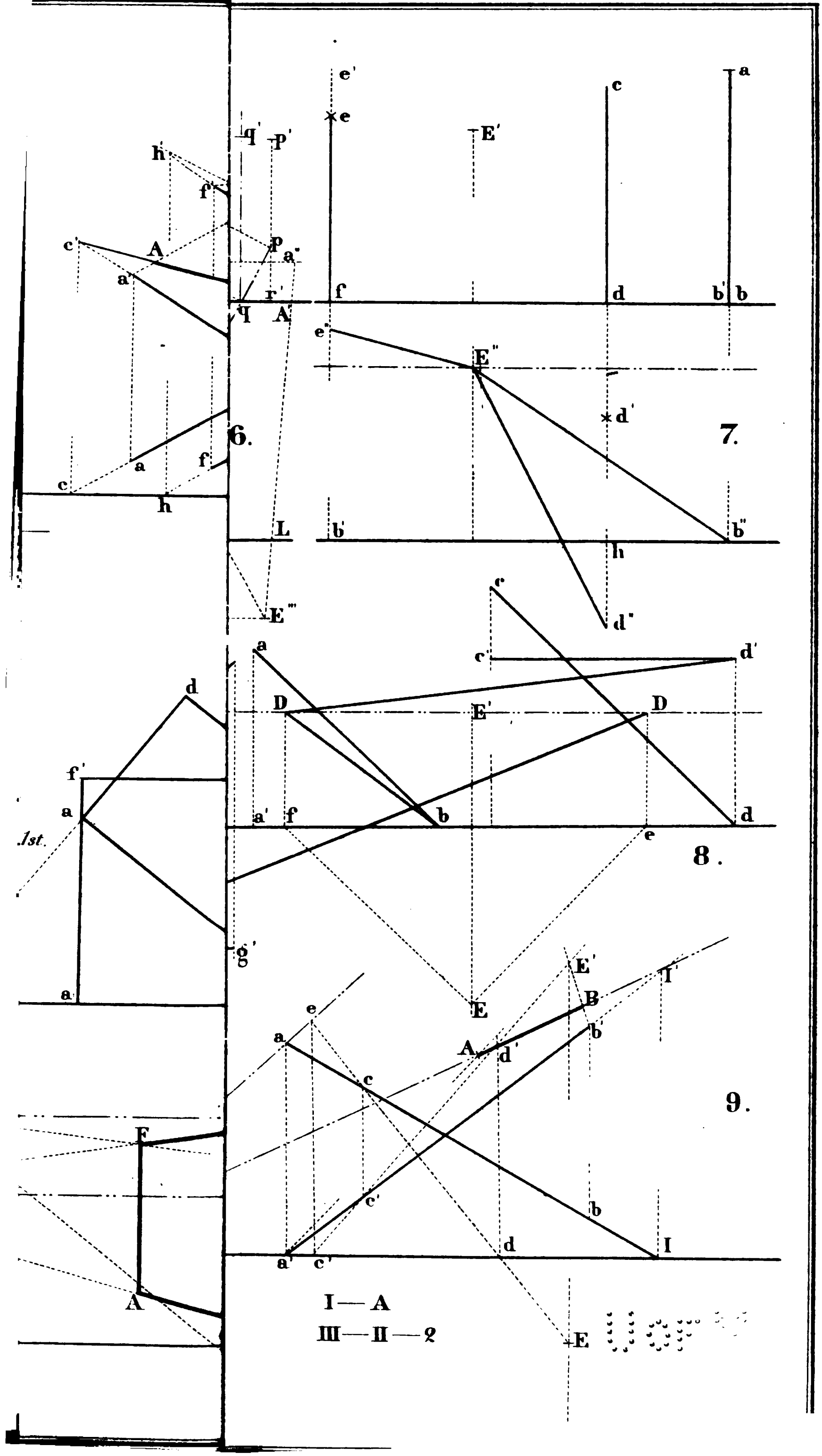
Here observe, that if any *plane* be perpendicular to the plane TT, its vanishing line will pass through K, the vanishing point of all *lines* perpendicular to the plane TT.

47. We will next, in returning to familiar methods, illustrate the foregoing principles in problems embracing the translation of the perspective plane, and the use of the vanishing points of lines, and the points in which they pierce the perspective plane, both of which have been proved (Th. III.) to be points of their perspectives.

Problem V.

To find the perspective of a straight line by means of visual rays; and with the perspective plane translated forward.

Let *ab—a′b′*, Pl. I., Fig. 3, be the given line, and E,E′ the point of sight. When the perspective plane is translated, it continues parallel to its first position, and carries the perspective figure with it, unchanged. Hence GL, being the ground line of the first position of the perspective plane, is its *real* position, and when it is translated to G′L′, the point of sight remains fixed, so that its horizontal projection, E, remains as


6.
7.
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before, and E″, its translated vertical projection, continues in the line E—E′, at a distance E″d″, above G′L′, equal to E′d′. In the details of the succeeding main step of the solution, two methods may be adopted, both of which are here illustrated.

First. Drawing the ray, aE—a′E′, we find its intersection with the real position of the perspective plane at A′, and immediately remove this point to A, making An=A′n′. The construction of A′ can then be erased to avoid confusing the projections. *Second:* b′ may be shown in translated position at b″; and bE—b″E″, the visual ray from it, pierces the perspective plane at B, the perspective of b,b′. This method, therefore, is equivalent to making the *vertical* projection of the given object only on the translated position of the perspective plane.

Problem VI.

To find the perspectives of two parallel straight lines, by means of their vanishing points and traces.

Let *ab—a′b′*, Pl. I., Fig. 4, be one of the given lines, having any position in space, behind the perspective plane, which stands on the ground line *cv*. This figure, being very simple, the perspective plane is not translated. EE′ is the point of sight. Then E*v*—E′*v*′, which pierces the perspective plane at *v*′, is the visual ray drawn parallel to *ab—a′b′*, and *v*′ is therefore the vanishing point of *ab—a′b′* and of all parallels to it.

The trace (36) of *ab—a′b′* is *c*′, another point of its perspective, and *c′v′* is the indefinite perspective of *ab—a′b′*. The perspective, *h′v′*, of any parallel to *ab—a′b′*, as *fg—f′g′*, joins *h*′, its trace, with *v*′ its vanishing point.

Visual rays from any points of *ab—a′b′* will meet the perspective plane in the indefinite perspective of the line. Hence, if we wish to find the perspective of the definite line *ab—a′b′*, we merely draw, *a*′E′, and *b*′E′, vertical projections of visual rays, and their intersections with *c′v′* give AB as the definite perspective of *ab—a′b′*.

Problem VII.

To find the perspective of a prismatic block, resting on the horizontal plane; by means of the vanishing points and traces of its edges.

Let *abcd*, Pl. I., Fig. 5, be the plan of the block, and E,E′ the point of sight. *qp* is the original, and *q′p′* the translated position of the perspective plane. Making $p''q''$, at a distance from $q'p'$ equal to $a'f'$, the height of the block, it will be the trace, on the perspective plane, of the plane of the top of the block, and will therefore contain the intersections, q'', o'', n'', and p'', of the top edges with the perspective plane. Also, v_1'', v_2'', at a distance from the ground line, $p'q'$, equal to the height of E′ above *pq*, is the horizon; which contains the vanishing points of all the top and bottom edges, since these are horizontal.

The rest of the solution is evident by inspection.

48. It is evident, on examination, *first*, that the edges, or elements, of a body can well be used as lines of construction, as just shown, only when they occur in a few sets of parallels; and, *second*, that, as seen at G, this method sometimes occasions intersections too acute for accuracy. The latter difficulty, and the similar one noted in (30) can often be avoided as follows: Translate a point, as *b,g*, Pl. I., Fig. 5, to the right or left, and parallel to the ground line, and find its perspective, G′ (not shown) by a visual ray. Then G will be on a parallel to the ground line through G′. This parallel will intersect some other line of construction, as the vertical projection of the visual ray from *b,g*, in a well-defined intersection.

49. It now remains to devise other systems of auxiliary lines, arranged in parallel sets. Two such sets are necessary, since in other methods than that by visual rays alone, every point of the perspective is found by the principle of (Th. II.) that *the perspective of any point in space is the intersection of the perspectives of any two lines through that point.*

50. The most generally convenient systems of auxiliary lines are those called *perpendiculars* and *diagonals*. A perpendicu-

lar is any line perpendicular to the perspective plane. A diagonal is any *horizontal* line, making an angle of 45° with the perspective plane.

51. By (33) the *vanishing point of all perpendiculars* is E′, the vertical projection of the point of sight, or the centre of the picture. Likewise, the *vanishing point of diagonals* must always be on the horizon (33) at a distance from E′ equal to the real distance of the point of sight from the perspective plane, shown by the distance of its horizontal projection, E, from the *original*, or *real*, position of the ground line.

52. The method of diagonals and perpendiculars may now be stated thus: Having any point whose perspective is to be found, pass a perpendicular, and a diagonal, through it, and find their perspectives. The intersection of these perspectives will be the perspective of the given point.

But the perspective of either a diagonal, or a perpendicular, is found by the method of Prob. VI., by joining its trace with its vanishing point.

Finally, as diagonals and perpendiculars are all horizontal, they will, if passed through a point in the horizontal plane, pierce the perspective plane in the ground line; and if passed through any other point, they will pierce the same plane as far above or below the ground line, as the point is above or below the horizontal plane.

53. The perspective of a straight line has already been defined (16) as the trace, on the perspective plane, of the visual plane containing the line. Accordingly, it is sometimes convenient to use *visual planes* as auxiliaries in construction, as well as auxiliary *lines* of any kind. Now, but one *visual* plane can be passed through a *line*, not containing the point of sight, and this plane will be determined by that line, together with a visual ray through any point of it. But any number of differently placed visual planes can be passed through a *point;* hence, if we find the perspective of successive points on a line, by means of visual planes, we can give these planes any desired position. A vertical one is generally most convenient. Indeed, as the trace on the *perspective* plane is the one which contains the perspective of the given point through which the visual plane is passed, no new method is really afforded by placing visual planes perpendicular to the perspective plane;

for the *vertical trace of such a plane* through any point, the *vertical projection of a visual ray*, from the same point, and the *perspective of a perpendicular*, through the same point, all coincide.

A vertical visual plane through a central point, or vertical line, of an object, is a *central* vertical visual plane.

54. By now adopting a suitable number and arrangement of planes of projection, and methods of construction, the perspective of any point, on any object, can be found. For among all the methods of construction, if one fails at a particular point another will conveniently apply.

55. In applying the last article to finding the perspectives of that large class of objects, such as many buildings, whose lines are arranged in a few sets of parallels, the method of using these lines themselves, and their traces and vanishing points, as in Prob. VII., should be freely used. *First.* To avoid crowding the figure with foreign lines of construction. *Second.* Because all parallels really do have a common vanishing point, but if, through instrumental errors of construction by other methods, they fail to have such a vanishing point in the drawing, the perspective becomes more disagreeably distorted than by any other ordinary error; since, in perspective, errors in the *direction* of lines offend the eye more than errors in their *length.*

56. To complete this introduction to the systematic general views, and detailed illustrations of the various particular methods of perspective, given in the next chapter, we here add separate constructions of the perspectives of diagonals and perpendiculars; a modification of the method of visual rays, as applied to a line as $b—b'g'$, Pl. I., Fig. 5; and an example of a visual plane passed through any line in space.

PROBLEM VIII.

To find the perspective of any perpendicular.

A perpendicular (50) may have either of these three different positions: it may be *in* the horizontal plane, *above* it, or *below* it. In either case, the centre of the picture (11) is its vanishing

point, and this, with its own trace (36) on the perspective plane, will (Th. III.) determine its perspective.

Now, Pl. I., Fig. 7, let *ab—b′*; *ef—e′*; and *cd—d′* have, respectively, the three positions named; let the perspective plane be translated forward from the original position *fb* to the position *hb″*, and let E′ be the primitive, and E″ the new position of the vertical projection of the point of sight or centre of the picture, these points being equidistant from their respective ground lines.

Then, remembering that a perpendicular to the vertical plane pierces that plane in the point which is the vertical projection of the perpendicular, *b″*E″ is the perspective of *ab—b′—b″*; *d″*E″ is the perspective of *cd—d′—d″*; and *e″*E″ is that of *ef—e′—e″*.

Remark.—Recollecting that the vanishing point is (31) *b″*E″, for instance, is the perspective of an infinite length of *ab—b′* from the perspective plane at *b* in the direction *ba;* and the like is true of any line.

Problem IX.

To find the perspective of any diagonal.

A diagonal (50) may have the same varieties of position as a perpendicular; and its perspective is likewise determined by its vanishing point and trace.

Now let *ab—a′b*, Pl. I., Fig. 8, be a diagonal in the horizontal plane; *cd—c′d′*, one at the height *dd′* above it; and *hg—h′g′* one at the distance *hh′* below it. If, then, E,E′ be the point of sight, draw the visual diagonal, E*f*—E′D′, which pierces the perspective plane at D′, which is therefore the vanishing point of *cd—c′d′* and of all parallel diagonals. Likewise E*e*—E′D determines D, the vanishing point of all diagonals parallel to *hg—h′g′*. Then as the three given diagonals pierce the perspective plane at *b*, *d′*, and *h′*, respectively, *b*D, *d′*D, and *h′*D′ are their perspectives.

Remark.—In this problem the perspective plane was not translated.

PROBLEM X.

To find the perspectives of points, by visual rays, when the projecting planes of those rays are nearly or quite perpendicular to the ground line.

Let the given points be the extremities of the vertical line, $a—a'b$, Pl. I., Fig. 6, and let E,E',E'' be the projections, before and after translation, of the point of sight. To find the traces of the visual rays, as $aE—a'E'$, in such a case revolve the vertical visual plane $Eb'a'$, containing it, about either of its traces, and into one of the planes of projection, when the supposed ray will be found in an available position. Thus by revolving the plane $Eb'a'$ to the right, about its horizontal trace Eb', till it coincides with the horizontal plane, the line $a—a'b'$ will appear at aa'', the point EE', at E''', and the visual rays from a,a' and a,b', at aE''' and $a''E'''$. Hence A' and B' are the revolved positions of the perspectives of a,a' and a,b'. Therefore, by making AB equal to $A'B'$, and Bk equal to $B'b'$, AB will be the perspective of $a—a'b'$ on the translated position of the perspective plane; whose ground line is kL.

EXAMPLE.—*Explain the construction, partially indicated, of the perspective of the vertical line $p'—r'p'$, Pl. I., Fig. 6, where the line itself is revolved as at pp''.*

PROBLEM XI.

To pass a visual plane through any straight line.

Any two intersecting straight lines determine a plane. If either of them be a visual ray, the plane will be a visual plane, passed through the other one. Hence, to solve this problem, pass a visual ray through any point of the given line. Its traces, together with those of the given line, will determine the traces of the visual plane.

The construction of Pl. I., Fig. 9, will now be evident.

CHAPTER II.

GENERAL TABLES AND FUNDAMENTAL PROBLEMS.

SECTION I.

General Tables.

57. By collecting, arranging, and expanding the results of the previous chapter, the following general tables and illustra-

Table I.

Of the number and position of the principal planes of projection used in perspective.

These planes may be	Two.	A.—Of which the vertical is also the perspective plane.	
	Three.	B.—Of which the perspective plane may be taken.	a.—Parallel to the vertical plane. b.—Perpendicular to the ground line. c.—Perpendicular to the central vertical visual plane.
		C.—Of which the horizontal *visual* plane is one, and used as a horizontal plane of projection.	
	Four.	D.—Of which the perspective plane is parallel to either vertical one of three co-ordinate planes of projection.	
		E.—The perspective and original (57) planes, and the visual planes, parallel to them.	

tive problems arise. They should be thoroughly studied and applied, since they embrace in a very condensed form the elements of every method of constructing the perspectives of objects. After due familiarity with them, their application in finding the perspectives of any real objects or structures, with their shadows or reflections, can be readily made. First in order comes Table I., on p. 23.

By an *original plane* (a term formerly used) is meant any plane, containing a figure whose perspective is to be found.

58. After choosing the number and real position of the planes of projection from the preceding table, and supposing the projections of any object to be given upon them, the next step, preliminary to the immediate construction of the perspective of that object, is to find its apparent contour. Hence follows—

Table II.

Methods of determining apparent contours.

Apparent Contours may be found on

- I. *Plane sided bodies*, usually by inspection, or by simply drawing a visual ray from a supposed point of contour.
- II. *Developable surfaces*, by drawing visual planes tangent to them, whose elements of contact will be those of apparent contour.
- III. *All other curved surfaces.*
  - *First:* By means of *tangent visual planes*, whose points of contact will be points of apparent contour.
  - *Second:* By *secant visual planes*, cutting sections from the surface, to which *visual rays* will be tangent at points of apparent contour.
  - *Third:* By *projections* of *tangent visual rays* on any meridian planes.

59. Having found the apparent contour of a body, its perspective can then be found. Hence the following—

Table III.

General methods of constructing perspectives.

Perspectives of points may be found—

- I.—By visual rays, given—
  - 1.—By their *projections* only.
  - 2.— " the *revolved ray itself*.
  - 3.— " a separate distorted *representation of the ray itself* (old method).
- II.—By points of concourse.
  - 1.—Taking *any pair of auxiliary lines*, through the point, the intersection of whose perspectives is the perspective of the point; as—
    - 1st.—*Lines of the object*, where in parallel sets.
    - 2d.—*Diagonals and perpendiculars;* or pairs of diagonals.
    - 3d.—*Lines, each having parallel projections.*
    - 4th.—*Verticals* or *parallels*, and *any other lines.*
    - 5th.—*Chords* of the angles between given lines and the perspective plane, with any other lines.
  - 2.—Using *any visual plane* with any *one auxiliary* line through each point, as—
    - 1st.—*Vertical visual planes* and *lines of the object.*
    - 2d.—*Vertical visual planes* and *diagonals.*
    - 3d.—*Vertical visual planes* and *perpendiculars.*
    - 4th.—*Vertical visual planes* and *verticals*, or parallels to the ground line, etc.
  - 3.—Using *any pair of visual planes* the intersection of whose traces on the Persp. Pl. will be the perspective of the point.
- III.—By co-ordinates.

60. The perspectives being now supposed to be found on the perspective plane in space, the final step is to clear the perspective from the projections. Accordingly here follows—

Table IV.

Methods of transposing the perspective plane, to avoid confounding the projections and the perspectives.

The perspective plane may be	In case of two planes	1°—*Translated forward* before revolution into the plane of the paper.
		2°.—*Translated backward* before such revolution.
		3°.—Revolved without translation, *provided the horizontal plane is revolved* 180°.
	In case of three or four planes	4°.—The perspective plane, *parallel to the vertical plane*, may be taken at such a distance from it as to be revolved directly back, without confounding itself with the vertical plane.
		5°.—*Translated to the right or left*, and then revolved about either trace into a plane of projection.
		6°.—*The perspective plane* translated *backward*, and the *visual horizontal plane* revolved *down* into it.
		7°.—In Table I., E, the two vertical planes are each revolved forward; and then the figure, Pl. XVI., Fig. 108, is revolved 180° around a vertical axis.

SECTION II.

Fundamental Problems.

§ 1.—*Apparent contours.*

61. As stated in Table II., the apparent contour (7) of a *plane sided body* can usually be determined by simple inspection. If the body is wholly in space, it may, however, be convenient to produce its faces till they intersect one or more planes of projection.

The apparent contour of a *developable surface* consists simply of the elements of contact of two tangent visual planes, the construction of which is very easy. Hence the finding of the apparent contours of these, and of plane sided bodies will

be sufficiently exhibited in the general problems of the following chapters, without separate treatment here.

The three cases of (III.) Table II., are, however, here illustrated, both for *convex* and *concave* surfaces, and with two, or three planes of projection. The problems are enunciated with direct reference to the notation in the tables employed in each. Thus, the following references are to Table I.—*Two* planes. Table II.—*Secant* visual planes. Table IV.—The perspective plane translated *forward.*

Problem XII.

To find points of the apparent contour of a sphere

by I.—A.
II.—III., *Second.*
IV.—1°.

Let the equal circles, with O and O′ for their centres, Pl. II., Fig. 10, be the projections of the sphere, and E,E″ the point of sight; after translation of the perspective plane forward. E*ab* is the trace on a horizontal plane O′*h*′, through the centre of the sphere, of a vertical secant visual plane, which cuts the sphere in a circle whose horizontal projection is *ab*. Revolving this plane into the plane O′*h*′, and to the left, E,E″ will appear at E‴ by making EE‴ perpendicular to the axis E*b* and equal to E″*h*′. The circle *ab* will appear as a circle, on *ab* as a diameter, and E‴T″ and E‴ *t*″ will be revolved positions of visual rays tangent to this circle at T″ and *t*″, which are therefore revolved positions of points of apparent contour. In returning the plane E*b* to its primitive position, these points revolve in arcs perpendicular to the axis E*b*, and appear at T and *t*; which are vertically projected at T′ and *t*′ by making T′*k*=TT″ and *t*′*r*=*tt*″. Therefore, T,T′ and *t*,*t*′ are points of apparent contour.

Any other secant visual plane, as E*d*, will give two other points.

Remarks.—a. Secant visual planes might also have been placed perpendicular to the vertical plane, and revolved into that or a parallel plane through the centre of the sphere, or revolved into or parallel to the horizontal plane. Also vertical

planes, as Eb, might have been revolved into or parallel to the vertical plane.

b. Other points of contour are easily found by *tangent* visual planes perpendicular to the planes of projection. Thus, the vertical plane EN gives the point N,n' on the horizontal great circle. Also E$''$P$'$, perpendicular to the vertical plane, gives the point P$'p$ on that great circle which is parallel to the vertical plane.

Example 1.—*Let E$'$ be below O$'h'$.*

Ex. 2.—*Let the perspective plane be translated backward.*

Ex. 3.—*Let the vertical secant planes be revolved into or parallel to the vertical plane of projection.*

Ex. 4.—*Let the secant visual planes be perpendicular to the vertical plane.*

Problem XIII.

To find points of the apparent contour of a piedouche by { **I.**—B—**b.** / **II.**—III., *First.* }

The piedouche, a little pedestal, is principally bounded by a concave surface of revolution. Let AB—A$'$B$'b'$, Pl. II., Fig. 11, be the piedouche, E,E$'$ the point of sight, and let the perspective plane (not shown) be perpendicular to the ground line.

The points of contour on the least horizontal circle, and on the meridian curve, parallel to the vertical plane of projection, none of which are shown, are readily found as in the last problem. The former points are in vertical tangent visual planes. The latter are in tangent visual planes perpendicular to the vertical plane.

To find intermediate points: Assume any horizontal section, $m'n'$—mtT, and make it the base and circle of contact of an auxiliary tangent cone whose vertex v,v' will therefore be in the axis, v—$a'b'$, of the piedouche. Two visual planes can be drawn tangent to this cone, and, at the intersection of their elements of contact with the circle of contact, $m'n'$, they will be tangent to the piedouche also, giving thus two points of apparent contour. Now these planes will be determined by the visual ray vE—v'E$'$, through the cone's vertex, and their traces on the plane m'P$'$, by the intersection of this ray with

lar is any line perpendicular to the perspective plane. A diagonal is any *horizontal* line, making an angle of 45° with the perspective plane.

51. By (33) the *vanishing point of all perpendiculars* is E′, the vertical projection of the point of sight, or the centre of the picture. Likewise, the *vanishing point of diagonals* must always be on the horizon (33) at a distance from E′ equal to the real distance of the point of sight from the perspective plane, shown by the distance of its horizontal projection, E, from the *original*, or *real*, position of the ground line.

52. The method of diagonals and perpendiculars may now be stated thus: Having any point whose perspective is to be found, pass a perpendicular, and a diagonal, through it, and find their perspectives. The intersection of these perspectives will be the perspective of the given point.

But the perspective of either a diagonal, or a perpendicular, is found by the method of Prob. VI., by joining its trace with its vanishing point.

Finally, as diagonals and perpendiculars are all horizontal, they will, if passed through a point in the horizontal plane, pierce the perspective plane in the ground line; and if passed through any other point, they will pierce the same plane as far above or below the ground line, as the point is above or below the horizontal plane.

53. The perspective of a straight line has already been defined (16) as the trace, on the perspective plane, of the visual plane containing the line. Accordingly, it is sometimes convenient to use *visual planes* as auxiliaries in construction, as well as auxiliary *lines* of any kind. Now, but one *visual* plane can be passed through a *line*, not containing the point of sight, and this plane will be determined by that line, together with a visual ray through any point of it. But any number of differently placed visual planes can be passed through a *point;* hence, if we find the perspective of successive points on a line, by means of visual planes, we can give these planes any desired position. A vertical one is generally most convenient. Indeed, as the trace on the *perspective* plane is the one which contains the perspective of the given point through which the visual plane is passed, no new method is really afforded by placing visual planes perpendicular to the perspective plane;

be drawn tangent to $B'd'$ produced, giving another point of contour, in the same manner.

Passing to the next topic, $E'd'$ is the vertical trace of a horizontal visual plane, containing the circle $Tt—d'h'$. The tangent visual rays, ET and Et, can therefore at once be drawn, giving T,T′ and t,t' for points of apparent contour.

Other secant visual planes would intersect the piedouche in curves, each of which would require a series of auxiliary planes by which to construct it. Hence, as before stated, only four points are readily found by this method.

Problem XV.

To find points of apparent contour on a piedouche,
by { I.—B—b.
II.—III., *Third.*

Continuing to refer to Pl. II., Fig. 12, AH is the horizontal trace of any meridian plane on which E,E′ is projected by the perpendicular EP at P. When the plane AH is revolved about the axis of the surface, A—$a'b'$, till parallel to the vertical plane of projection, P appears at P″, and in vertical projection at P‴, since EP is in the horizontal plane Ed'. Then P‴u''' is the revolved position of the projection of a visual ray upon the plane AH, and $u'''u''$ the revolved position of a point of contour, and uu', found as was cc' in the last problem, is its true position. To prove this, it is only necessary to consider, 1*st*, that the point of contact of a tangent visual plane is a point of apparent contour; 2*d*, that a tangent plane to a surface of revolution is perpendicular to the meridian plane through the point of contact; 3*d*, that, therefore, the point of contact of the trace of the tangent plane on the meridian plane, with the meridian curve, is the point of contact of the tangent plane; but 4*th*, that this trace is by (2*d*) the projection upon the meridian plane of any line in the tangent plane. Hence u,u', which, by construction, is the point of contact of the projection of a visual ray on the plane AH, is the point of contact of a tangent visual plane, and is therefore a point of apparent contour as required.

Any number of points of contour between cc', the lowest and the highest, can be found by this method, as well as by that of auxiliary tangent cones or spheres, as explained in Prob. XIII.

§ 2.—*Perspectives of points and straight lines.*

62. To afford the readiest means of comparing methods of solution in the following problems, one and the same magnitude, similarly placed, also, in nearly every case, will be given to be put in perspective. That magnitude will be a definite straight line, one end of which shall rest on the horizontal plane, and the other shall be in space.

63. This being understood, the problems will still be enunciated in the briefest manner by reference to the notation of methods found in the preceding tables.

Thus, to find the perspective of a straight line by

$$\begin{cases} \textbf{I.}\text{—A.} \\ \textbf{III.}\text{—II., } \mathbf{1}\text{—}2d. \\ \textbf{IV.}\text{—1°.} \end{cases}$$

means: To find its perspective by the method, Table I. (A), of *two* planes; using Table III. (II., **1**), *pairs of auxiliary lines*, which shall be (2*d*) *diagonals* and *perpendiculars:* and with the perspective plane, Table IV. (1°), *translated forward*, before revolution back into the horizontal plane. The number of the table in every case is the one first given.

It will not be necessary to give examples of every combination of methods which the preceding tables would afford. Enough, however, will be given, and sufficiently varied, to enable the student to make use of any combination of methods that he pleases.

64. Table III. being of chief importance, the order of the following problems will be, principally, that of the methods as noted in the final subdivisions of that table.

The easy apprehension of the solutions will be greatly promoted by observing that the same letter remains attached to the same point in all its translated and revolved positions, and that the required perspective of the point is indicated by the same letter, capital.

The perspective of a straight line, by

$$\begin{cases} \textbf{I.}\text{—A.} \\ \textbf{III.}\text{—I—}\mathbf{1}. \\ \textbf{IV.}\text{—1°.} \end{cases}$$

has already been found in Prob. V.

Problem XVI.

To construct the perspective of a straight line, by

$$\begin{cases} \text{I.—B—b.} \\ \text{III.—I—1.} \\ \text{IV.—5°.} \end{cases}$$

Let $ab-a'b'$, Pl. II., Fig. 13, be the given line, E,E′ the point of sight, and PQP′ the position of the perspective plane, separate from, and perpendicular to the two planes of projection, whose ground line is QQ′.

Also observe, that here, and in every case of the use of three planes, the eye, at E,E′, is supposed to be looking perpendicularly towards the perspective plane PQP′.

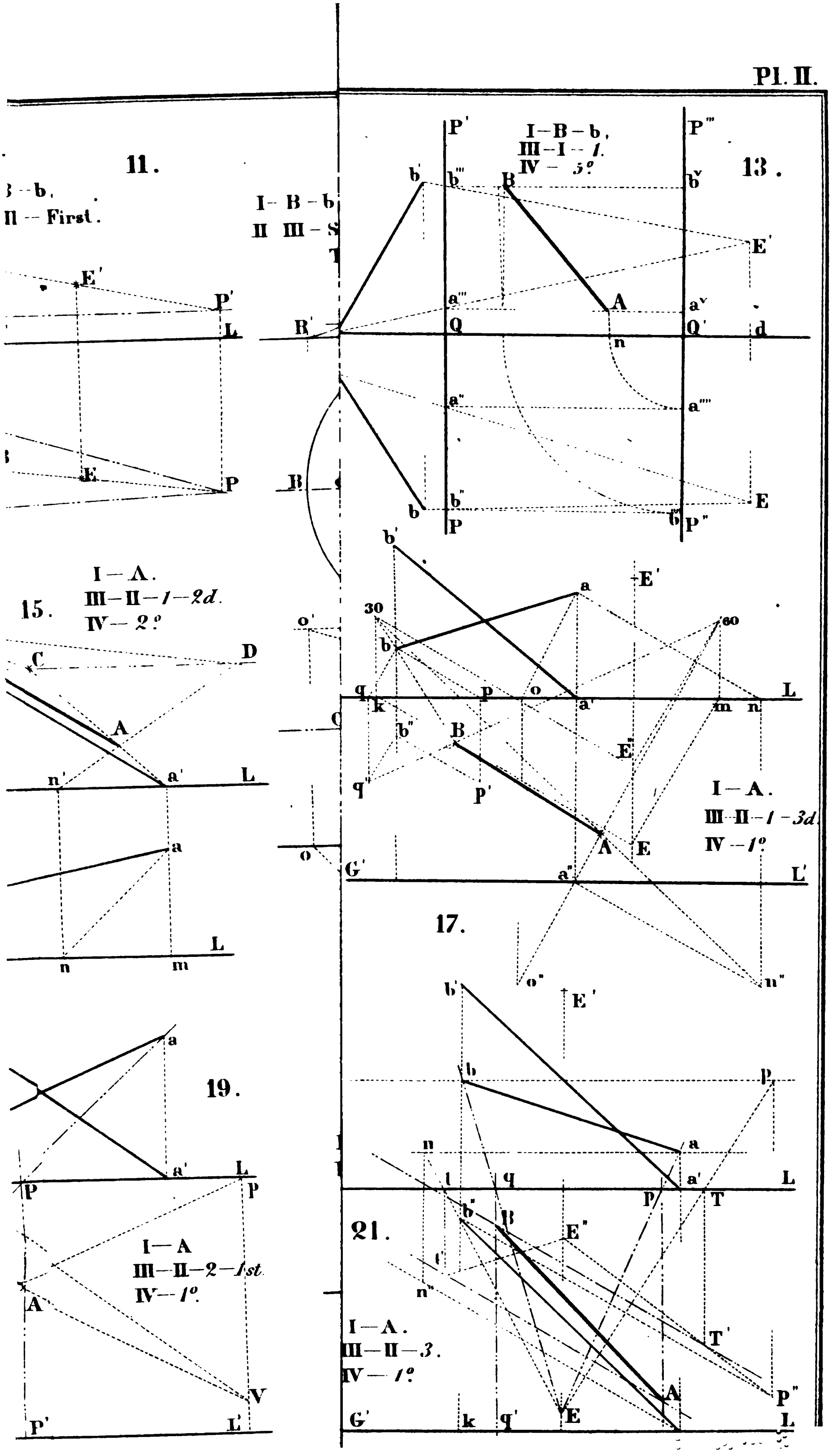
Now, $a\text{E}-a'\text{E}'$ is the visual ray from the point a a' in the horizontal plane, and $b\text{E}-b'\text{E}'$ is the ray from the point b, b' in space. The former ray pierces the perspective plane in a'',a''', the two projections of the perspective of a,a'. In translating the perspective plane, parallel to itself, to the second position, P″Q′P‴, the point $a''a'''$ proceeds in the line $a''a''''-a'''a^{\text{v}}$, parallel to the ground line, to the position $a''''a^{\text{v}}$. Thence, in revolving P″Q′P‴ to the left, about its vertical trace, Q′P‴, as an axis, the latter point revolves in the horizontal arc, $a''''n-a^{\text{v}}\text{A}$, to A, the required perspective of aa'.

Likewise the visual ray, $b\text{E}-b'\text{E}'$, pierces the perspective plane at $b''b'''$, which, after translation and revolution, as before appears at B, the perspective of bb'. Therefore AB is the required perspective of $ab-a'b'$.

The perspective plane must here be revolved only to the left, for bb' is the left hand point of the given line to the observer at E, as he looks in the direction, Q′Q, and this revolution brings B at the observer's left, as he looks perpendicularly at the plane of the paper.

The projection of EE′, on the revolved position of P″Q′P‴, can easily be found. It will evidently be on a horizontal line through E′ and at a distance to the left of Q′P‴ equal to Ed.

Remark.—The above very detailed explanation is here given to be thoroughly learned, so as to render an abridged statement intelligible in future applications of the same method.


11.
B -- b.
II -- First.
13.
I—B—b.
III—I—1.
IV—5°
I— B — b
II III — S
15.
I—A.
III—II—1—2d.
IV—2°
17.
I—A.
III—II—1—3d.
IV—1°
19.
I—A
III—II—2—1st.
IV—1°.
21.
I—A.
III—II—3.
IV—1°

EXAMPLE.—Let EE′ be to the left of PQP′, and *ab*—*a′b′* on the right of PQP′.

PROBLEM XVII.

To find the perspective of a straight line, by

I.—B—c.
III.—I—1.
IV.—5°.

The solution can be sufficiently understood from Pl. II., Fig. 14. Prob. VII. represents the case of

$$\begin{cases} \text{I.—A.} \\ \text{III.—II—1—1}st. \\ \text{IV.—1°.} \end{cases}$$

PROBLEM XVIII.

To find the perspective of a straight line by

I.—A.
III.—II—1—2*d*.
IV.—2°.

Here, Pl. II., Fig. 15, *ab* is the horizontal projection of the given line, and GL the original ground line of the vertical or perspective plane, two planes only being used. Translating this plane backwards to G′L′, and revolving it into the plane of the paper, the vertical projection of the given line appears at *a′b′*, the centre of the picture (11) at C, and the horizon, through C, parallel to G′L′, at CD.

The horizontal projection of the eye not being given, D may be assumed anywhere on CD, for the vanishing point of diagonals (51) C is the vanishing point of perpendiculars. Then *an*—*mn* is the diagonal from the point *aa′* in the horizontal plane. It pierces the perspective plane in *n*, which is translated along *nn′* to *n′*, and *n′*D is the perspective of this diagonal. *am*—*m* is the perpendicular from *a*,*m*. It pierces the perspective plane at *m*, which is translated at *a′*, and *a′*C is the perspective of *am*. Hence A, the intersection of the perspectives of the diagonal and the perpendicular through *aa′*, is the perspective of *aa′*.

Again, remembering that diagonals and perpendiculars (50) are always horizontal, $bo—b'o'$, which pierces the perspective plane at o', is the diagonal from bb'. Therefore o'D is its perspective. Likewise $bd—b'$, the perpendicular from bb' pierces the same plane at b', and b'C is its perspective, intersecting o'D at B, the perspective of bb'. Hence AB is the required perspective of $ab—a'b'$.

Remarks.—a. It is now evidently unnecessary to draw *am* and *bd*. Also, knowing the height of bb', we have only to make $o'g$ equal to that height to find o', and the vertical projection of the given line is unnecessary; and the like is true for any number of points of known height, so that *sixteen* lines will give AB, by this problem.

b. Only one perpendicular can be drawn through a given point, but two diagonals can be so drawn.

Problem XIX.

To find the perspective of a straight line by

I.—A.
III.—II—1—2*d*.
IV.—3°.

Here, Pl. II., Fig. 16, instead of translating the perspective plane, the horizontal plane is revolved 180°, and the real position of *ab* is, accordingly, as far behind the ground line as its revolved position is now in front of it.

Let CD be the horizon, and C and D respectively the vanishing points of perpendiculars and diagonals. The method of solution from Table III. is the same as in the last problem. Thus, D*n* is the perspective of the diagonal $an—a'n$, and a'C that of the perpendicular $aa'—a'$, hence A is the perspective of aa'. Likewise o'D is the perspective of the diagonal $bo—b'o'$, and b'C, that of the perpendicular, $bd—b'$; hence B is the perspective of bb', and AB of $ab—a'b'$.

Remarks.—a. In this method, observe, that the vanishing point of a given revolved diagonal, as *an*, is the opposite to the one which would be taken for a diagonal *in the same direction* behind the ground line. To find unmistakably the vanishing point of a given diagonal by the present method, construct the

horizontal projection of the point of sight, which, on the revolved horizontal plane, will be on a perpendicular, through C to the ground line oL, and at a distance *behind* it equal to CD. Then by (Prob. IX.) we could find the vanishing point of a given diagonal, as an—$a'n'$, by finding the trace (6) on the perspective plane of a visual ray parallel to it.

b. The method of this construction requires fewer lines than any other. Thus there are but fifteen lines of all kinds in Fig. 16, while in Fig. 13 there are twenty-one, and in some as many as thirty. But there may often be some points to which any given method will not apply advantageously. We therefore proceed with the following methods:

PROBLEM XX.

To find the perspective of a straight line by

I.—A.
III.—II—**1**—3*d*.
IV.—1°.

This method, Pl. II., Fig. 17, consists in using auxiliary lines, whose parallel projections make some angle with the ground line which is afforded by one of the triangles.

Thus, let ab—$a'b'$ be the given line, EE′ the point of sight, and GL the ground line, in their real position, and let EE″ and G′L′ be the two latter after translation forward. The translated position of a' is a''; then let an—$a''n''$ be an auxiliary, through aa'', whose projections each make an angle of 30° with the ground line. Its trace (6) is n'', and by (36–37) and drawing the parallel visual ray Ek—E″30, 30 is its vanishing point, hence n''30 is its perspective. Again, ao—$a''o''$ is a second auxiliary, whose projections make an angle of 60° with the ground line. Its trace is o'', its vanishing point at 60, the trace of the parallel visual ray Em—E″60, and o''60 is its perspective. Hence A, the intersection n''30 and o''60 is by (35) the perspective of a,a',a''.

Observing that b'' is the translated position of b', the student may trace out the similar construction of B, the perspective of $bb'b''$, remembering also that auxiliary lines through this point,

parallel to those through $aa'a''$ will have the same vanishing points as the latter.

Problem XXI.

To find the perspective of a straight line by
I.—A.
III.—II—1—4*th.*
IV.—1°.

A vertical line, used here as an auxiliary, has no trace on the perspective plane, and (33) no vanishing point. Hence the perspective of some point in it must be found by some other method than that of Theor. III. *One* point will suffice, however, since the direction (17) of the perspective is known. Therefore in Pl. II., Fig. 18, we shall first find the perspective of b', the foot of a vertical line through bb'. Using the method of diagonals and perpendiculars, with D and E′, the centre of the picture, for their respective vanishing points, we find B′ for the perspective of b, and B′—B for that of the vertical line $b—b'q$ through bb'. For the second auxiliary through bb' take a diagonal bp, which by (Prob. IX.) has p'' for its trace, $p''p'$ being equal to $b'q$. Then p''D is the perspective of bp and B of bb'.

To apply this method to aa', the perspective of some other point of a vertical line through it must be found, as of a,d'; the vertical projection, d', being translated to d'', making $a''d''= a'd'$. But aa' being in the horizontal plane, it is so easy to find its perspective directly that we have done so by two methods. *First.* By diagonals and perpendiculars, whose perspectives n'D and a''E′ intersect at A, the perspective of aa'. *Second.* By 30° and 60° *horizontal* auxiliary lines $ak—a''k'$ and aL—a''L′, whose vanishing points, found as shown by (Prob. VI.), are the points 30 and 60 respectively. Hence k'30 and L′60 are the perspectives of ak and aL, and they intersect at A the same perspective of aa' as before.

Remark.—The method of finding B is of frequent use in finding the perspectives of points nearly level with the eye, and when the auxiliary vertical line happens also to be a line of the given object itself.

Problem XXII.

To find the perspective of a straight line by

I.—A.
III.—II—2—1*st.*
IV.—1°.

This is the method by vertical visual planes, which is convenient, since it occasions but few lines; and less apparently artificial than the method of diagonals and perpendiculars, since it employs both projections of the point of sight.

Let $ab-a'b'$, Pl. II., Fig. 19, be the given line; GL, the first, and G'L' the second position of the perspective plane, and E,E'' the projections of the point of sight, after translation of the perspective plane. Then aE is the horizontal, and PP' the vertical trace of a vertical visual plane through aa'. The vertical trace, PP', therefore contains the perspective of aa', which will be, definitely, at the intersection of the perspective of any auxiliary line, through aa', with PP'. The present method requires the given line itself to be the auxiliary. Accordingly we find its trace (6) I', observing that b'' is the translated position of b'. Also by (32) we find its vanishing point V, and thus have its perspective I'V, which intersects PP' at A, the perspective of A.

The same line V.I' meets NN', the vertical trace of a vertical visual plane through bb', and whose horizontal trace is bE, at B, the perspective of bb'. Then AB is the required perspective of bb'.

From (53) we see why the auxiliary visual planes are never taken perpendicular to the vertical plane.

Problem XXIII.

To find the perspective of a straight line by

I.—A.
III.—II—2—2*d.*
IV.—3°.

Here again, Pl. II., Fig. 20, the horizontal plane is revolved, and the perspective plane not translated. Let $ab-a'b'$ be the

given line, EE′ the point of sight, its horizontal projection, E, being behind the ground line after revolving the horizontal plane. DE′ is the horizon, and E′D=*e*E by (51) D being the vanishing point of diagonals. Then, *a′e* being the ground line, *a*E is the revolved horizontal trace, and NA the vertical trace of a vertical visual plane through *aa′*. Also, *an*—*a′n* is the diagonal through *aa′*, whose perspective, *n*D, is determined as usual by its trace, *n*, and vanishing point D. Now *n*D intersects NA at A, the perspective of *aa′*.

The construction of B is now evident from inspection.

Problem XXIV.

To find the perspective of a straight line by

I.—A.
III.—II—**3.**
IV.—1°.

This method, Pl. II., Fig. 21, is of less value for practical convenience than to exemplify the general principle that the intersection of the traces, on the perspective plane, of *any* two visual planes through a point is the perspective of that point.

Let *ab*—*a′b′* be the given line, GL the first, and G′L′ the translated position of the ground line, EE″ the point of sight, and *a″b″* the vertical projection of the given line after this translation. Then a visual plane through *bb″* will be determined by *any line*, *bp*—*b″p″*, through *bb″*, and a visual ray through *any point*, *pp″*, of that line; etc.

Problem XXV.

To find the perspective of a straight line in the horizontal plane by

I.—C.
III.—II—**1**—1*st*, 2*d*, 5*th*.
IV.—6°.

The method of Table I., here exhibited, being somewhat peculiar, it is more fully illustrated than the previous methods,

being employed in connection with various methods from Table III. in this and the next two problems.

L*t*, Pl. III., Fig. 22, is the original, and L′T the translated position of the ground line. DV is the horizon, and E′ the vertical projection of the point of sight. Now the plane of the horizon may be taken as the horizontal plane of projection, *for all the lines drawn through the eye in determining all the necessary vanishing points.* If this plane be revolved about the horizon as an axis, into the perspective plane, both planes may then be revolved in the usual way about the ground line, and into the plane of the paper, taken as the principal horizontal plane of projection.

The horizon plane may be revolved either way about DV, but it is preferable to revolve it downwards, so as to make the visual rays in it, to the vanishing points of given lines, appear parallel to those lines.

The point of sight *itself* will then appear at E, at a perpendicular distance, EE′, from DV, equal to the real distance of the eye from the perspective plane.

We have next to recall two principles: *First*, the projections of the same line, or of any parallel lines, upon one or more parallel planes, will be parallel; *Second*, when a line is in a plane, it coincides with its projection on that plane.

Accordingly, now, EV, parallel to *ab*, a given line in the horizontal plane, determines V as the vanishing point (32) of *ab*. Also, making E′D=E′E, we have D as a vanishing point of diagonals.

Now to apply the three methods referred to in the enunciation:

First. T translated from *t* is the trace of *ab* on the perspective plane, and V is its vanishing point. Hence TV is its indefinite perspective. The given object in this instance being but a single line, the other methods must be introduced to find the perspectives of its limits, *a* and *b*. Hence,

Second. Draw the perpendicular *ao*, which pierces the perspective plane at *o*, a point translated at O. Hence OE′ is the perspective of this perpendicular. The diagonal *an* finds its trace at *n*, which travels to N, and NO is its perspective. Therefore A is the perspective of *a*. Thus A is found, if TV be not drawn. Otherwise, the perpendicular, or the diagonal, alone, will suffice.

Third. With t as a centre, and radius tb, draw the arc bu, or merely make $tu=tb$, and draw the chord bu. Then btu will be an isosceles triangle, in which $bt=ut$. But VD is parallel to tu, and VE to bt, hence making VE″=VE, the chord of EE″ will be parallel to bu, which will make E″ the vanishing point of bu and of all parallels to it. Hence translate u, the trace of bu, to U, and UE″ will be the perspective of bu, and will intersect TV at B, the perspective of b.

Remarks.—*a.* Having gone thus far with these elementary methods, some abbreviations may be noted which can be made more or less fully in the previous figures, but which, if attempted too soon in practice, tend to confusion and error in the constructions. Produce ba to t, and draw ao. Then make $on=oa$, and make $tu=tb$, then TO=to; TN=tn, and TU=tu; also E′D=EE′, and VE″=VE. Then the actual drawing of an, of the chord or arc, bu, and of VE, and the arc ED or EE″ may be dispensed with. The student may reconstruct the figure thus abridged, and may usefully seek all possible abridgments in the previous figures.

b. The method *by chords*, just explained, does not require the peculiar arrangement of planes just shown. It may be applied to the previous figures, in which two planes are used, simply by representing both projections of the point of sight.

Problem XXVI.

To find the perspectives of various horizontal and vertical lines, as in the last problem, that is, by

I.—C.
IV.—6°.

In Pl. III., Fig. 23, are given ab, the horizontal projection of a line in the principal horizontal plane, and of a line vertically over it, at the height TT′ above it; also c and f, the horizontal traces of vertical lines, whose heights are equal to TT′.

First. When ab is *in* the horizontal plane, A, the perspective of a, is the intersection of the perspective TV of tb, with ND the perspective of the diagonal an. Also E″, as well as D, being found as in the previous figure, B, the intersection of

lar is any line perpendicular to the perspective plane. A diagonal is any *horizontal* line, making an angle of 45° with the perspective plane.

51. By (33) the *vanishing point of all perpendiculars* is E′, the vertical projection of the point of sight, or the centre of the picture. Likewise, the *vanishing point of diagonals* must always be on the horizon (33) at a distance from E′ equal to the real distance of the point of sight from the perspective plane, shown by the distance of its horizontal projection, E, from the *original*, or *real*, position of the ground line.

52. The method of diagonals and perpendiculars may now be stated thus: Having any point whose perspective is to be found, pass a perpendicular, and a diagonal, through it, and find their perspectives. The intersection of these perspectives will be the perspective of the given point.

But the perspective of either a diagonal, or a perpendicular, is found by the method of Prob. VI., by joining its trace with its vanishing point.

Finally, as diagonals and perpendiculars are all horizontal, they will, if passed through a point in the horizontal plane, pierce the perspective plane in the ground line; and if passed through any other point, they will pierce the same plane as far above or below the ground line, as the point is above or below the horizontal plane.

53. The perspective of a straight line has already been defined (16) as the trace, on the perspective plane, of the visual plane containing the line. Accordingly, it is sometimes convenient to use *visual planes* as auxiliaries in construction, as well as auxiliary *lines* of any kind. Now, but one *visual* plane can be passed through a *line*, not containing the point of sight, and this plane will be determined by that line, together with a visual ray through any point of it. But any number of differently placed visual planes can be passed through a *point;* hence, if we find the perspective of successive points on a line, by means of visual planes, we can give these planes any desired position. A vertical one is generally most convenient. Indeed, as the trace on the *perspective* plane is the one which contains the perspective of the given point through which the visual plane is passed, no new method is really afforded by placing visual planes perpendicular to the perspective plane;

UE″, the perspective of the chord *bu*, with TV, is the perspective of *b*.

Second. When is *ab* at the height TT′. Draw T′R′ the vertical trace of the horizontal plane containing the line, and all the horizontal auxiliaries used with it. Then T′R′ will contain the traces, on the perspective plane, of all these lines. Thus, T′V is the indefinite perspective of the upper line *ab*; N′D, of the upper diagonal *an*, and A′, of the point over *a*, at the height TT′. But A′ is also the intersection of the vertical line AA′ with T′V, or with the perspective of a perpendicular, diagonal, chord, or any line through the supposed point over *a*.

Likewise B′ will be the intersection of the perspectives of *any* two lines through the point *b*, whose height equals TT′. In the figure it is shown as the intersection of T′V with the vertical line BB′, or with U′E″, the perspective of the chord *bu* in the plane T′R′.

Third. Passing to *f*, its perspective is found at F by a diagonal *fr* and perpendicular *f,k*, whose perspectives are RD and KE′. Then F′ is found at the intersection of R′D, the perspective of the upper diagonal *rf*, with the vertical line FF′, perspective of the given vertical line at *f*. Finally the verticals at *f* and *c* being of equal height, and equidistant from the perspective plane, C and C′ may be found by drawing lines, as F′C′, parallel to the ground line, and noting their intersections with AB and A′B. Also, C′ may be found at the intersection of Q′E″, the perspective of a chord *cq* in the plane T′R′, with A′B′, and then C can be found by drawing the vertical line C′C.

Remark.—If, in finding F, the diagonal *fs* had been used, it would have been necessary to have found its vanishing point, which would have been at the right of E′ at a distance equal to E′D.

Problem XXVII.

To find the perspective of any straight line by

I.—C.
III.—II—1—*5th*.
IV.—6°.

Here, Pl. III., Fig. 24, we return to the general case of an oblique line, *ab*—AO, resting on the horizontal plane at one

end, a,A. Let aL be the first, and AL′ the second position of the ground line, DV the horizon or ground line of the second horizontal plane, E,E′ the point of sight in this plane, D the vanishing point of diagonals, and E″ of horizontal chords. Also let OF be the vertical trace of a horizontal plane, containing the upper point of ab and the foot of the parallel line fg—Fg'; and let $g'k$ be the vertical trace of a horizontal plane containing gg'. Then EV—E′V′ is the visual ray parallel to ab—AO, and piercing the perspective plane in V′, which is therefore the vanishing point of ab—AO, and of all parallels to it. Now A is the translated trace a, of ab, and F, likewise, the trace of fg—Fg', hence AV′ and FV′ are, respectively, the indefinite perspectives of ab—AO and fg—Fg'. All horizontal auxiliaries through the point bO have their vertical traces in OF, and all like lines, through gg', have theirs in kg'. Hence ND is the perspective of the diagonal bn—ON, and UE″ is that of the chord bu—OU. Either of these lines limits AV′ at B, the perspective of b,O. Again, g'E′ is the perspective of the perpendicular through gg', and it meets FV′ at G, the perspective of gg'.

65. The remaining methods, from Olivier, and illustrative of perspective as an abstract problem of descriptive geometry, may be better apprehended as such, after familiarity with the preceding, in which the natural facts of vision are more obviously recognized. They are also, in part, arranged to illustrate the three leading divisions of Table III.

66. The perspective of a *point* has been defined, in reference to natural facts, as the intersection of a visual *ray* with the perspective plane. That of a straight *line*, as the trace of a visual *plane* upon the perspective plane. But as an abstract problem, the original point or line is a given one, the point of sight is also a given one, and the perspective plane is given. Hence the abstract problem of the perspective of a point is this: *To find where a line, joining two given points, pierces a given plane.* And of the perspective of a line it is this: *To find where a plane, containing a given point and a given line, intersects a given plane;* that is, the perspective plane, which is separately shown.

Problem XXVIII.

To find the perspective of a point by

I.—B—a.
III.—I—1.
IV.—4°.

AA′, Pl. III., Fig. 29, the original point, and *pp′* the point of sight, are the two given points, and PQ the given (perspective) plane, QL being the ground line of the vertical plane. *Ap*—A′*p′*, the line joining the given points, pierces the perspective plane at *aa′*. Then *a′*, the vertical projection of the perspective of AA′, after revolving the vertical plane backward into the plane of the paper, is the perspective point practically required. Or, if the perspective plane PQ be revolved directly back, *a′* will appear at a distance above *a* equal to *a′h*.

Problem XXIX.

To find the perspective of a point by

I.—B—a.
III.—II—3.
IV.—4°.

Let AA′, Pl. III., Fig. 27, the original point in the horizontal plane, and *pp′*, the point of sight, be the two given points, and B*q* the given (perspective) plane. AB and AC are the horizontal traces of any two planes, each containing the point AA′. These traces intersect the perspective plane at B and C, whose vertical projections are B′ and C′ in the ground line. As these planes are also to pass through *pp′*, they will be fully determined by the lines *pq*—*p′q′* and *pr*—*p′r′*, through *pp′* and parallel to the horizontal traces AB and AC; and the intersections, *pp′* and *qq′*, of these lines with the perspective plane, will be other points of the intersections of the auxiliary planes with the perspective plane. Hence B*q′* and C′*r′* are the vertical projections of these intersections, and their intersection, *a*, is the (vertical projection of the) perspective of AA′, since two planes through AA′ and *pp′* must intersect in a line

joining these points, and this line must pierce the perspective plane in the intersection of the traces of the two planes on that plane.

Observe that, as the perspective plane, Bq, is parallel to the vertical plane of projection, the vertical projection of the perspective, by this method, is equal to the perspective itself in the plane Bq.

Problem XXX.

To find the perspectives of any parallel lines by

I.—B : **a.**
III.—II—1—1*st.*
IV.—4°.

Let LL′, L_1L_1', and L_2L_2', Pl. III., Fig. 25, be three parallel lines oblique to both planes of projection; and let *pp* be the given point (of sight). The perspective plane is PQ, and GT is the ground line of the vertical plane of projection. The perspective of each line is determined by an auxiliary plane, passed through it and the point *pp′*, and each of such planes is determined by the given line in it, and by a parallel, *pn—p′n′* through *pp′*. The construction will now be evident from Theor. I. and Theor. III.

Problem XXXI.

To find the perspective of a point by

I.—B : **a.**
III.—II—**3.**
IV.—4°.

Let AA′, Pl. III., Fig. 28, the original point in space, and *pp′* the given point (of sight) be the given points; and BP the given (perspective) plane. AB—A′B′ and AC—A′C′ are a pair of diagonals through AA′. Also AD—A′ is a perpendicular through AA′. Planes, through any two of these lines and the point *pp*, will intersect the plane BP in lines whose intersection with each other will, by the last problem, be the perspective of AA′. The rest of the solution may be understood by inspection.

PROBLEM XXXII.

To find the perspectives of parallels to the ground line by
I.—D.
III.—III.
IV.—4°.

Let LL'—L_1L_1'—L_2L_2', Pl. III., Fig. 26, be the given parallels, GT the ground line, POR an auxiliary vertical plane, and PQ the given (perspective) plane, and let $pp'p''$ be the three projections of the given point (of sight).

We evidently have m,m',m'' as the three projections of the perspective of L,L',L'', of which m' only is practically considered, it being the same as the perspective itself.

The ground of the title "method of co-ordinates" now appears in that instead of points of concourse, L, for instance, is co-ordinated at L'' by $L''p$ and $L''h$, and at L—L', by Op and Ok; and in that its perspective, likewise, is co-ordinated at m'', by pm'' and em'', and at m—m', by OP and On.

PROBLEM XXXIII.

To find the perspective of a point by
I.—D.
III.—III.
IV.—4°.

The planes, Pl. III., Fig. 29, being as in the last problem, let A,A',A'' and p,p',p'' be the given points, located as shown by co-ordinates from the three axes having O for the origin. PQ is the (perspective) given plane. Now a vertical plane through A,A',A'' intersects the plane PQ in the vertical line a—ha', and a plane parallel to the ground line through the same point intersects the same plane in the line aQ—a''—$a'n$, parallel to the ground line. These lines, as appears at ha' and na', are co-ordinates and locate a,a',a'' the perspective of A,A',A'', when $pp'p''$ is the point of sight.

67. The methods yet remaining to be exhibited, viz., those of

$$\begin{cases} \text{Table I:—E.} \\ \text{" III=I.—3.} \\ \text{" IV:—6°.} \end{cases}$$

are mainly of historical interest, and to qualify one to read and understand the earlier works on perspective. They can be summarily explained with sufficient fullness from Pl. XVI., Fig. 108, and the few following figures.

68. In Pl. XVI., Fig. 109, the modern names of most of the points and lines are written against them; and those not now used have their names in quotation marks.

The old names for these lines and points are generally as follows:

GL=*intersection line;* and, generally, the vertical trace of any plane is called its intersection line.

EM=*seat of the* visual ray M*e*.

*e*E′=*direct radial*, or direct visual ray.

*e*V=*radial of* MN., or parallel visual ray to MN.

MM′=*direct ray*.

M′=*seat* of M.

RR=*the original plane*, whether perpendicular or oblique to the perspective plane HGL.

HGL=the *picture*, or *plane of projection*.

*e*DD′=the *directing plane*.

*e*HV′=the parallel of the original plane RR.

KV′=the *visual projection*, or simply the *projection* of MK.

69. Next, comparing Figs. 108 and 109, Pl. XVI., we see that the vertical planes HGL and *e*DD′, and the horizontal plane *e*HV′ are supposed to be hinged at their intersections, so as to turn forward all together without change of relative position among the lines on any one of them, till they all coincide with the horizontal or original plane RR. Then Fig. 109, which is lettered exactly to correspond with Fig. 108, is to be viewed in the direction of the arrow. M*e*, then, is a sort of irregular or abnormal, yet true representation of the actual ray, M*e* in space, Fig. 108, as will next be proved.

THEOREM V.

The representative, after revolving the perspective plane, of an actual ray through a given point, passes through the perspective of that point, and hence is an available line of construction.

In Fig. 108, observe the similar triangles, MQM″ and ME*e*, whose sides, M″Q and *e*E, are parallel. Now, as the axes GL and DD′, about which the latter lines revolve in passing to the position shown in Fig. 109, are parallel, M″Q and *e*E will remain parallel, and without change of length, as at QM″ and E*e*, Fig. 109. Also the side ME remains fixed, hence, in Fig. 109, also, MQM″ and ME*e* are similar triangles, so that M*e* is still a straight line through M″, though it represents, in a distorted manner, the ray itself, as distinct from its projections ME and M′E′. Hence M*e* is, as stated, available as a line of construction.

Remark.—Recalling from (42 : 6°) that the planes RR and *e*HV′ merely show a particular application of the principle that *any* plane, and the *visual plane through its vanishing line* are parallel, every line of construction in Fig. 108 would have its corresponding line in the general case in which the parallel planes RR and *e*HV′ should make any angle with the perspective plane. But MM′ and *e*E′ would no longer be projecting lines, GL would not be the ground line, but simply the vertical trace of RR; nor would HV′ be the horizon, but simply the vanishing line of RR, and E′, the vanishing point of MM′, and separate from the vertical projection of the point of sight.

70. We now see, by inspection of Figs. 108 and 109, that there are the following ways of constructing the perspective of a point M, equally applicable whether RR is a horizontal or an oblique plane.

First. M″ is the intersection of the *visual ray*, M*e*, with M′E′, the trace of a visual plane through MM′.

Second. M″ is the intersection of M*e* with QM″, the trace, on the perspective plane, of a visual plane through E*e*, the perpendicular from the eye to the directing line DD′.

Third. M″ is the intersection of M*e* with KV′ or NV, the perspectives of any line MK (or MN) through M, in the plane

RR, this perspective being determined by the trace K (or N) and vanishing point V′ (or V) of the line MK (or MN).

Fourth. M″ is the intersection of M′E′ with QM″, perpendicular to GL.

Fifth. M″ is the intersection of M′E′ with either KV or NV, found as in (Third).

Sixth. M″ is the intersection of QM″ with either KV′ or NV.

Seventh. M″ is the intersection of KV′ with NV.

Eighth. M″ is the intersection of NV—found by making it parallel to *e*N′ and through N—with any of the preceding lines. This is called the method by directors.

To favor a clearer understanding of these methods we will next give separate illustrations of a few of them, first supposing RR to be a horizontal plane.

Problem XXXIV.

To find the perspective of a straight line lying in any horizontal plane by

I.—E.
III.—I—3.
IV.—6°.

Using the same notation as in Pl. XVI., Fig. 109, let M*m*, Fig. 3, be the given line, GL the ground line, HE′ the horizon, E′ the centre of the picture, and *e* the representation of the eye itself. Then M*e* represents the visual ray itself from M, and M′E′, its vertical projection on the perspective plane, or which is the same thing (53) the perspective of the perpendicular MM′. Then by (70 *First*) M″, the intersection of M*e* and M′E′, is the perspective of M. In like manner, *m*″ is the perspective of *m*, and hence M″*m* is the perspective of M*m*.

Here we see the horizontal projection of the point of sight is not required. In the next problem it is.

In some of the older works the ground line GL is omitted.

Problem XXXV.

To find the perspective of a straight line as in the last problem, but by (70 *Second*).

Let M*m*, Fig. 4, be the given line, and let the other lines be as before, as shown by their letters.

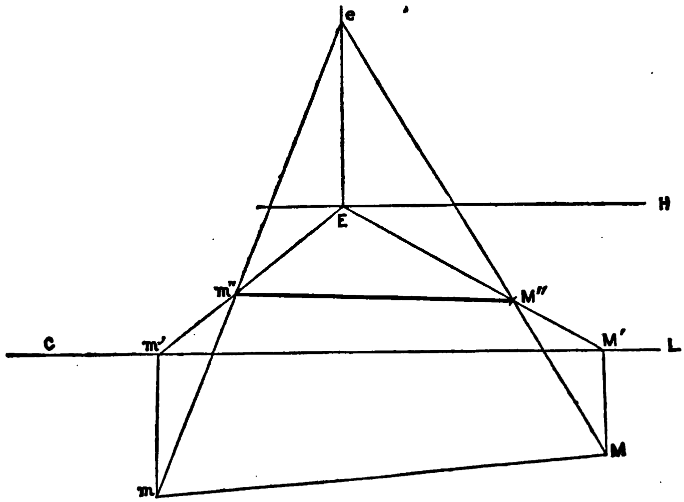


FIG. 3.

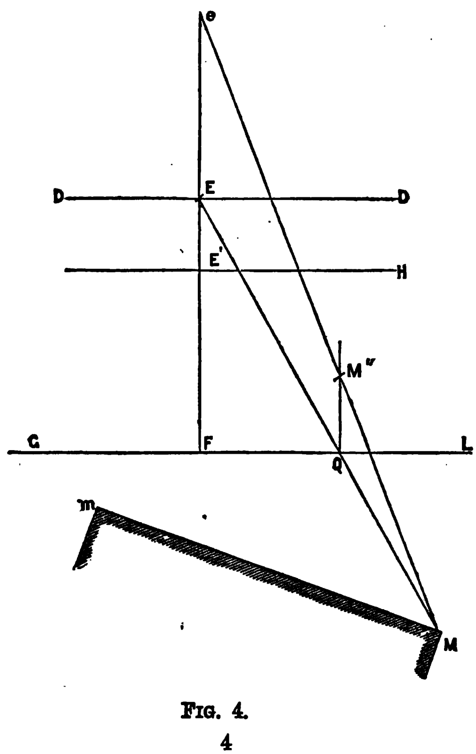


FIG. 4.

Observe that if Mm be one side of a house, for example, represents the side towards the eye, that is the front, to an eye at e facing the front of the house.

Here make EF, Fig. 4, equal to eE′, and E, on the directing line DD′, will be the horizontal projection of the eye, represented by e. Then draw MQE, the horizontal projection of the ray Me, and QM″, perpendicular to the ground line GL, will meet Me at M″, the perspective of M.

The student can in like manner find the perspective of m.

Next, to compare the methods in which Me is used with those in which only the true projections of e are employed, let us immediately take the following case.

Problem XXXVI.

To find the perspective of a straight line as before, but by (70 *Fourth*) *that is, by a visual ray given only by its projections.*

Let Mm, Fig. 5, be the given line, the other given parts being understood from their letters, and FE being equal again to eE′, to find the horizontal projection, E, of the point of sight.

M′ being the vertical projection of M, we have ME and M′E′ as the two projections of the visual ray from M, and this ray, by the usual construction, pierces the perspective plane at M″, which is therefore the perspective of M.

The perspective of m can be found without further explanation.

To facilitate the comparison of methods, the ray itself, Me, is added, to show that it also passes through M″.

Hitherto the given line, Mm, has been taken in the horizontal plane. Let it now be in a different plane.

71. The earlier treatises being made before descriptive geometry was known, explain nothing of the method of finding the traces of a plane given, as planes commonly are, by three points or two lines in it, or by a line in it, and the inclination of the plane to a plane of projection; hence they assume the trace of a plane upon the perspective plane to be known, together with its inclination to that plane.

At present, the construction of the traces of a plane, under the foregoing conditions, is a very simple and familiar operation. We therefore proceed to illustrate the early methods, finally, by their use in finding the perspectives of two lines in planes which are given by their traces on the perspective plane, and their inclination to it.

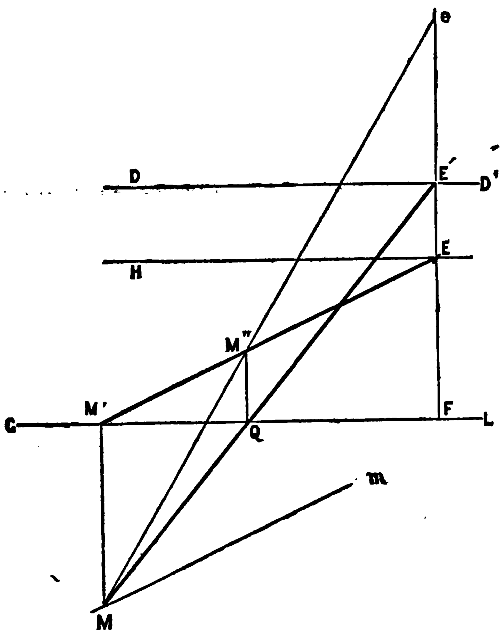


FIG. 5.

PROBLEM XXXVII.

To find the perspective of lines in any plane by

I.—E.
III.—I.—3.
IV.—6°.

1°.—*The perspectives of the lines.*

Let AB, Pl. XVII., Fig. 110, be the vertical trace of a plane, E the vertical projection of the point of sight, and EE″ the horizon.

E*e* is the vertical trace of a vertical visual plane perpendic-

ular to the perspective plane. By revolving this plane into the perspective plane, E will appear as at E″, by making EE″ equal to the known distance of the eye from the perspective plane. Then make EE″C equal to the complement of the known inclination of the given plane, AB, to the perspective plane, and draw E″C, and, through C, draw MN, parallel to AB, for the vanishing line of the given plane. Likewise, by Prob. III., find PQ the vanishing line of a plane *ab*, whose inclination to the perspective plane is EC′E′.

Now both of the given planes being supposed to be revolved to coincide with the perspective plane, AD and FG represent given lines—one in the plane AB, the other in *ab*. Their intersections, at A and F, with the traces AB and *ab*, are their intersections with the perspective plane, and, consequently, are points in their perspectives.

Again: from *e*, the representative of the eye, draw *e*N, parallel to AD, and *e*′H, parallel to FG—having made C′*e*′=C′E′. These are the visual rays, which, being parallel to the given lines, determine their vanishing points, which are at N and H, respectively, since they must be in the vanishing lines of the planes containing the given lines. Thus AN is the perspective of AD, and HF is that of FG.

2°.—*The perspective of the intersection of the given planes, and thence the intersection itself.*

K, being the intersection of the vertical traces of the two planes, is the point where that intersection pierces the perspective (or vertical) plane, and hence is one point of its perspective. But M, the intersection of the vanishing lines of the two planes, is the vanishing point of their line of intersection, hence MK is the perspective of that intersection.

Again: the visual ray from the vanishing point of a line, is parallel to the line. Hence M*e* is thus parallel to its original, which is KL, the intersection of the two given planes represented on the plane AB. A line through K, parallel to M*e*′, would be the representation of the same intersection on the plane *ab*; both AB and *ab* being still understood to be revolved about their traces, AB and *ab*, into the perspective plane.

A line through K, and parallel to ME, would evidently be the vertical projection of KL upon the perspective plane. This will be more evident by constructing a figure similar to Fig. 112, but in which the plane UAB should ascend from the

eye, *e*, so that PDE should slant downwards toward the right.

Finally: notice, in this figure, that MK and AN intersect on EE″, also that MK and FH intersect on EE′, the relative horizon of the plane *ab*. This indicates that AD pierces the horizontal visual plane, EE″, in the intersection of the two given planes, AB and *ab*; and that FG likewise pierces the visual plane EE′ (parallel to the plane *ab*) in the intersection KL (whose perspective is KM) of the two given planes. But to show that this is wholly accidental, let A*d* be a new line in the plane AB. The parallel ray *en* gives its vanishing point *n*, and *n*A, the perspective of A*d*, does not intersect MK on the horizon.

CHAPTER III.

Perspectives of Forms only.

72. The preceding full exhibition of the various *general* methods of finding contours and perspectives, is, at the same time, quite *abstract*, an isolated straight line, or a point, having been taken for illustration, for the sake of *brevity* and exclusive attention to the *principle* of each construction.

Therefore, to avoid confusion among so many methods, we will, before introducing a new body of examples, proceed with a review and application of the preceding problems in others of a practical character; in which, however, for the sake of simplicity, the shadows will be omitted. And we will begin this chapter with a practical comparison of the early and present methods of solution of problems, involving lines and planes situated in any manner in space.

PROBLEM XXXVIII.

Having given the vertical trace of a plane, its intersection with a second plane, and the angle between the two planes, to find the vertical trace of the second plane, its vanishing line, and the perspective of the intersection.

1°.—*The vanishing line of the given plane, and the revolved position of the intersection of the planes.*

Let the perspective plane, taken as the vertical plane, Pl. XVII., Fig. 111, be the plane of the paper. Let AB be the vertical trace of the given plane, E″E′ the horizon, containing E′, the centre of the picture, and let E′CE″ be the inclination of the plane AB to the perspective plane. Then make E′E″ equal to the distance, supposed to be known, of the eye from the perspective plane, and make E′E″C equal to the complement of the inclination just described, and CV, drawn through C and parallel to AB, will be the vanishing line of the plane AB.

Pl. XVII., Fig. 112, illustrates this problem pictorially, and is hence lettered to correspond with Fig. 111. ABE′F′ is the perspective plane, ABEI the given plane, receding downwards from the eye at *e*, and CV is its vanishing line.

Next let the intersection of the plane AB, with the second plane mentioned, be given by its vertical projection P′N. We will first find its position, after revolving the plane AB to coincide with the perspective plane. DQ, parallel to E″C, is the revolved position of the intersection of the part of the plane AB behind the perspective plane, with a plane perpendicular to the trace AB on the line PE′. Hence, drawing P′Q parallel to AB, we have DQ for the true length of P′D. Then, making DP=DQ we have PN itself, the same as PN in Fig. 112, only after revolution about AB into the plane of the paper, that is, the perspective plane.

2°.—*The vertical trace of the second plane.*

Now let HRT, both figures, be the angle, revolved about RT into the paper, in Fig. 111, between the two planes, R being anywhere on PN, and RT being perpendicular to PN. It is then evident that a line HY, in space parallel to PN, is a line in the second plane, and, therefore, that it determines Y, in the perspective plane, as a point of the vertical trace of this plane.

To use HY, in Fig. 111, conceive a plane through H, parallel to the plane AB, and at a perpendicular distance from it equal to HT. Then make DT′=T″T, and T′H′ perpendicular to DQ and equal to TH, and H′O, parallel to DQ, will be the intersection of the parallel plane with the perpendicular plane PE′. Hence OY, parallel to AB, is the trace of the parallel plane, and GH″=OH′ is the true distance of H from this trace; GH″ being drawn through T, perpendicular to AB, since the plane HGTT″, Fig. 112, is perpendicular to the line AB, that is, to both the perspective plane and the plane ABU. Hence draw H″Y parallel to PN, and YN will be the required vertical trace of the second plane.

Otherwise: Pass a plane, RK*g*, through R, parallel to PDE′, and in counter revolving R to its true position in space, its vertical projection will be R′, whose perpendicular distance from the perspective plane is R′*r*, found by drawing K*r*, parallel to DQ, since the parallel planes PDE′ and RK*g* cut parallel lines from the same plane, AB, and by then drawing R′*r* par-

allel to AB. Then make rh=TH, as before, and hg, parallel to H′Q, will meet RKg at g, a point of the trace OY; after finding which, we proceed as before.

3°.—*The vanishing line of the second plane.* Two methods are here given for finding this line.

First Method. Make Ce=CE″ and e represents the eye, as explained on Pl. XVI., Fig. 108. Then eV, parallel to PN, gives V as the vanishing point of the intersection PN, since, as this line is in the plane AB, its vanishing point must be in the vanishing line CV of that plane. Then by (43) VL, parallel to NY, is the vanishing line of the second plane.

Second Method. At any point, P′, of the vertical projection of the intersection of the planes, conceive two planes, each perpendicular to the trace of one of the given planes. The perpendicular, from P′ to the perspective plane (similar to RR′, Fig. 112, for the point R′, Fig. 111), is common to these two planes, which contain the angles made by the given planes with the planes of projection. Hence make P′Q′=P′Q and make P′S perpendicular to YN, and Q′SP′ will be the angle made by the second plane with the vertical plane. Hence, finally, make E′M perpendicular to NY, and E′E‴=E′E″ and make E‴ME′=Q′SP′, and M will be a point of the vanishing line of the second plane, through which ML is drawn parallel to NY.

Thus E‴M is parallel to Q′S′, the angle E‴ME′ is in front of the perspective plane, and Q′SP′ is behind it, to the eye at e.

Finally, V being the vanishing point of the intersection PN, and N being its intersection with the perspective plane, NV is its perspective.

V could have been found, evidently, by drawing E′V, the vertical projection of a visual ray, parallel to NP′, the vertical projection of the given intersection.

Problem XXXIX.

To construct the required parts of the last problem, by the method of two fixed planes of projection.

1°.—*The traces of the second plane.*

Let Qm, Pl. XVII., Fig. 113, be the ground line, and let SM and DN′ be the traces of any oblique plane parallel to it,

and inclined *downwards* and forward, contrary to Figs. 111 and 112, merely to avoid the confusion of constructions in the second angle.

Let SN—S'N' be the given intersection of the plane SM—DN', with the second plane, which is determined by this intersection, and the given angle, $=x$, which it makes with the first plane.

S is one point of the horizontal trace of the second plane, and N' is a point of its vertical trace. Now revolve the projecting plane, SNN', about its horizontal trace SN, and into the horizontal plane, and NN''=NN'; and N''S is the revolved position of the intersection of the two planes. Then make any point, K, the revolved vertex of the given angle, x, between these planes, and draw KH perpendicular to N''S, to give H, a point in the horizontal trace of the plane of this angle, since the latter plane is perpendicular to the line of intersection of the given planes. Then HM, perpendicular to SN, is the horizontal trace of the plane of the angle, and by making HK'=HK and drawing K'M, we get K'M for one side of the angle x, when revolved about HM into the horizontal plane. Then make MK'O$=x$, and O, the intersection of K'O with HM, is a point in the required horizontal trace of the second given plane. Hence SOR is that trace, and RN' is its vertical trace.

If the plane, SM—DN', had been given by its vertical trace, and inclination to the vertical—which is the perspective—plane, as in the last problem, the given inclination would have been first shown, as at QDP'. Then by a counter-revolution of that triangle, Q would return to P, a point in the trace SM.

Also let only the vertical projection S'N' of the intersection of the given planes, be the given one. Then p', one point of this intersection, would appear, after revolution, in DQ at q', by drawing $p'q'$ parallel to the ground line. In counter revolution, q',q returns to p',p, and N and p would determine NS before finding SM.

2°.—*To construct the vanishing lines*, etc.

E and E' being the projections of the eye, make E,E'' the position of the eye, after revolution about the trace P'C of the vertical visual plane PP'C. Then draw E''C, to make E''CE'= QDP', and C, being the trace of a visual ray parallel to the given plane, SM—DN', the line CV, parallel to the horizon E'm will be the vanishing line of SM—DN'.

Now, the visual ray, whose horizontal projection is En, being parallel to the intersection SN—S′N′, is a line of the parallel visual plane whose trace is CV, hence it pierces the perspective plane in CV at V, the vanishing point of the intersection of the given plane, and hence a point in the vanishing line Vm' (parallel to the trace N′R) of the second plane.

Otherwise, Em—E′m' is the visual ray parallel to the horizontal trace, SR, of the second plane, and it meets the perspective plane in the horizon at m', a point of m'V, parallel to RN′. Vm' is thus determined, *either* by the two points V and m', or by either of them together with the direction of Vm', parallel to RN′.

Finally, the perspective, N′V, of the intersection SN—S′N′, joins its trace, N′, with its vanishing point V, which is evidently the intersection of the vanishing lines of the two given planes.

Remark.—In comparing this problem with the last, the perplexing nature of the method of one plane of projection there used will prevent surprise at the following grave error in a standard work in which the methods of Pl. XVI., Fig. 109, and Pl. XVII., Fig. 111, prevail. It says, make DO=DP=DQ draw a line through O, perpendicular to NP′, and equal to HT, and through its extremity, a line parallel to P′N, to meet OY at Y; a construction which would wrongly locate Y, and for which no reason could be given.

THEOREM VI.

If any two curves, or a straight line and a curve, are tangent to each other, their perspectives will be tangent.

First. For the visual cones, from the given curves, having a common vertex, will be tangent to each other along an element; hence the curved intersections of these cones with any plane will be tangent; but these intersections are the perspectives of the given curves; hence the proposition is proved.

Second. If one of the given curves be replaced by a tangent straight line, its visual cone would become a plane, and its perspective, a straight line tangent to the intersection of the perspective plane with the remaining cone. That is, *if a*

straight line be tangent to a curve, their perspectives will be tangent. Thus, let ATB, Pl. VII., Fig. 62, be any curve, and TD a tangent to it at T, and let E be the point of sight. The visual rays from the curve collectively form a cone, and those from the tangent, a tangent plane along the element E T. Then any plane, PQ taken as a perspective plane, will cut a line *td* from the tangent plane, tangent to the curve *atb* cut from the cone. The line *td* is the perspective of TD, as *atb* is of ATB, and their point of contact, *t*, is on TE, and is therefore the perspective of T.

Third. The last demonstration shows also that *the perspectives of the bases of cones and cylinders, are tangent to the perspectives of their extreme visible elements.* For the visual planes, through these elements, are tangent to the visual cones from the bases, hence their traces, which are the perspectives of those elements, will be tangent to the traces of the cones, which are the perspectives of those bases.

Problem XL.

To find the perspective of a vertical cylinder.

1°.—*Description of parts and perspective of the lower base.* Let the cylinder, Pl. IV., Fig. 30, stand on the horizontal plane, and be tangent to the perspective plane, with AEH for its horizontal projection, and let NQ, the first position of the ground line, be translated backwards to *nq*. By using diagonals and perpendiculars, as is here done, or *any horizontal* lines of construction, the necessity for a vertical projection will be avoided, and the vertical trace, *n′q′* merely, of the plane of the upper base will be sufficient.

Let DD′ be the horizon, D and D′ the vanishing points of diagonals, and C the centre of the picture.

The perspective of the cylinder will now consist of the perspectives of its two bases, and of its extreme visible elements. The element, A—*aa′*, of contact with the perspective plane, will appear in its real size.

To find a point in the lower base, the point W for example. The perpendicular, W*d*—*d″*, through this point, has J for its original and *d* for its translated trace on the perspective plane,

and C is its vanishing point; therefore *d*C is its perspective. WP—*dp* is the diagonal from E, its trace P is translated to *p*, and *p*D′ is its perspective. Therefore *w*, its intersection with *d*C, is the perspective of W. In like manner we find the perspective of any other points of this base, taking eight equidistant points, usually in practice, as seen in the figure, and which are commonly sufficient.

2°.—*Particular points of the lower base.*

Drawing the perpendicular, OA, and diagonal, OJ, from the centre of the base, *o*, the intersection of their perspectives, is the perspective of the centre O. Having found *a*, the perspective of A, and *g*, that of G, the line *ag* will be the conjugate axis of the ellipse which is the perspective of the lower base. It results then, that *s*, the *centre of the perspective*, differs from *o*, *the perspective of the centre* O. To find the point of which *s* is the perspective, proceed in an inverse order. Thus, D′*sr* is the perspective of the diagonal through the point sought, then translate *r* to R and draw a diagonal RS, and S will be the point whose perspective is *s*. KL, parallel to the ground line and through S, is that chord whose perspective is *kc*, the transverse axis of the perspective ellipse, and the elements standing on K and L are the extreme visible elements of the convex surface.

The last conclusion verifies what was known in advance, viz., that from any point of sight, at less than an infinite distance, less than half of the convex surface of a cylinder would be visible. Here, KAL is the horizontal projection of the visible segment, and this is further verified by observing that tangents at K and L would properly meet at a distance before the ground line, NQ, equal to CD.

3°.—*Tangents to the perspective of the base.*

The tracing of other curves than circles is greatly aided by knowing, beforehand, several tangents to them. We will therefore point out the tangents to the base *aek*, which can be readily found according to (Th. VI.).

The perspectives, *n*C and *d*C, of the perpendiculars EN and WJ; the perspectives, as *q*D′ of the tangent diagonals at H,B,F and T; the ground line, and the perspective of the parallel to it at G, are eight tangents to *aek*.

The perspective of the parallel at G is parallel to the ground line, through *g*. Thus having ten points, eight of which are

points of contact with known tangents, and four of which are extremities of the axes, we can sketch the elliptical perspective of the base very accurately.

4°.—*To find any point in the perspective of the upper base.* Take the point whose horizontal projection is F, for example. The vertical projection and the trace of the perpendicular, FU, through this point is u'; and u'C, is its perspective; remembering that $u'q'$ is the vertical trace of the plane of the upper base. Likewise $u'd'$ is the vertical projection of the diagonal FJ, and d'D is its perspective, giving at f', its intersection with u'C, the perspective of F, in the upper base.

In the same manner, all the points of the perspective of the upper base can be found, also the perspective of the centre, and the centre of the perspective of this base can be found, as for the lower base. Each tangent, as qD′ to the lower perspective, has a corresponding one, as q'D′ to the perspective of the upper base, and any point, as t, in the lower base, finds a corresponding upper point, t', the intersection of an element as tt', with the diagonal, as d'D′, or a perpendicular over those through t.

After thus finding the perspectives of the bases, draw vertical common tangents to them, as at k and c, the extremities of a transverse axis, and the perspective of the cylinder will be complete.

Remark.—Observe that the diagonals OF and OH are those of a circumscribing square of the circle AEK, and that each contains two points, as B and H, from which perpendiculars are also drawn; also that of these perpendiculars, as Bu, each contains two points as B and F, so that, by these lines, perspective points can be rapidly found. But by using the *tangent* diagonals, as above, the perspective ellipses can be rather more accurately sketched.

Problem XLI.

To find the perspective of a circle contained in a plane perpendicular to the ground line.

1°.—*The perspectives of points of the circle.*

This construction, Pl. IV., Fig. 31, is of use in finding the

perspectives of round-topped windows in the side walls of buildings. Let FG—H′L′ be the given circle, Fp the original, and DE the translated ground line be also the horizon, making the eye to be in the horizontal plane, with E for its vertical projection, and D for the vanishing point of diagonals. Operating, then, by the method of diagonals and perpendiculars, Hn is the horizontal projection of the diagonals through the highest and lowest points of the given circle. H′n'' and L′n' are their respective vertical projections, and n''D and n'D their perspectives. In like manner HF—H′ and HF—L′ are the perpendiculars from the same highest and lowest points, and H′E and L′E are their perspectives, which intersect n''D and n'D, respectively, at h'' and h', the perspectives of the points H,H′ and H,L′. In the same manner other points of the perspective are found.

The circle being tangent to the perspective plane at FF′, that point is its own perspective. H′L′qr is the perspective of the circumscribing square, and therefore affords, as seen, four tangents to assist in sketching the ellipse.

2°.—*The perspectives of tangents to the circle.*

In the last problem, eight tangents to the perspective ellipse were readily found. *To find, then, four more tangents to the present perspective ellipse.* For this purpose we need lines in the plane of the circle, and tangent to it at points analogous to B,F, etc., in Fig. 30. Then revolve the plane of the given circle about its vertical trace F—FH′, and F′H″L″ will be the revolved position of the circle, and F′,H″,G″, and L″, the points whose tangents have already, as at H′E, been put in perspective. A diameter D″C″, bisecting F′H″, and the chords D″A″ and B″C″, parallel to the ground line, determine the required original points. The tangents at these points are analogous, in the plane of the circle, to diagonals, and hence may be called co-diagonals. Their vanishing points X and Y are therefore found by making EX=EY=ED. Furthermore, they intersect the perspective plane, in the trace of the plane of the circle, at K,M,P, and N (the latter not shown). Hence their perspectives, which are tangent by (Theor. VI.) to that of the circle, are respectively KX, MY, PX, and NY.

73. It is evident, on inspection, that the axis of the perspective ellipse is not vertical. It can only be so, when the eye is in a horizontal plane through the centre of the circle; this

relation being analogous, for a circle perpendicular to the ground line, to that of the horizontal circle and point of sight as seen in Fig. 30.

74. Two radii of concentric circles will include parallel chords, which, being parallel, will have a common vanishing point. Recollection of this principle will assist in finding the perspectives of such circles. Moreover, it will be obvious on making the construction, that *the perspectives of the concentric circles will not be concentric ellipses.* For that segment of the longer radius, which is between the two circles, and nearest to the eye, will appear larger than the opposite and more remote similar segment, ending at a point as G, Pl. IV., Fig. 30.

EXAMPLE 1.—*Find the perspective of a cylinder whose axis is parallel to the ground line.*

Ex. 2.—*Do. when the axis of the cylinder is parallel to the horizontal plane only.*

Ex. 3.—*Do. of a circle lying in a plane oblique to both planes of projection.*

PROBLEM XLII.

To find the perspective of an inverted cone.

Let V′H′T′, Pl. IV., Fig. 32, be the vertical projection of the cone, and let the horizontal plane be revolved 180°, giving the circle, with radius VT, for the horizontal projection of the upper base, the circle VS for that of the intersection of the cone with the horizontal plane, that is, its trace, and V for that of its vertex; all these horizontal projections being as far, before revolution, behind the ground line as now in front of it. Let the method of diagonals and perpendiculars be again resorted to; but, for further variation from the last problem, let the horizon be taken at DE′, above the upper base, and E′, the centre of the picture, to one side of a plane through the axis, and perpendicular to the ground line.

1°. *The perspectives of the bases.*

These are found as in the last problem, only, as the eye is not in a plane perpendicular to the ground line through the centre of either base, the axes of these perspectives cannot be so readily constructed as in the former case. The peculiarities of this

case will be further discussed in a future problem. Find, therefore, the perspective of any convenient number of points of each base, as in the last problem, and join them. Thus, to find the perspectives of S and F, in the cone's trace, draw the perpendiculars, SS′, and FB′, whose perspectives are S′E′, and B′E′, and draw the diagonal FSp, whose perspective is pD. These perspective lines intersect at s and f, which are the perspectives of S and F. Passing to the points A and T, for example, in the upper base, the vertical projections and traces of the perpendiculars, AB′ and Tu, are respectively, n' and u', and their perspectives are n'E′ and u'E′. The vertical projection of the diagonal, whose horizontal projection is ATm, is $n'm'$, and m'D is its perspective. Dm' intersects n'E′ and u'E′ at a and t, the perspectives of A and T.

As in the last problem, eight tangents to the perspective of each base can be found by using the tangent perpendiculars and diagonals, and the parallels to the ground line, tangent as at F and C. Observe that as the perspectives of the latter lines are the parallel tangents at c and f, the line cf is a diameter of the ellipse csf. The like is true of ba, at the extremities of which, tangents parallel to LL would be the perspectives of tangents at A and B.

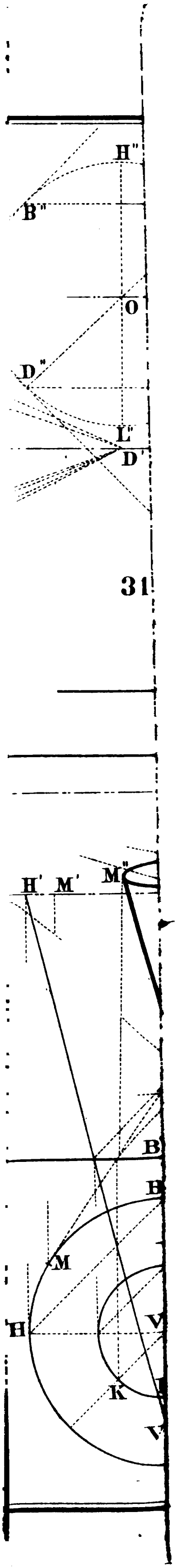
2°. *The perspective of the vertex.*

The vertex is taken below the horizontal plane, to give an opportunity of applying a perspective construction to a point in the third angle. In full, VB′ is the horizontal, and V′ the vertical projection of the perpendicular through the vertex VV′. Then V′ is its trace, and V′E′ its perspective. Also, Vr is the horizontal, and V′r' the vertical, projection of the diagonal through the vertex, and r'D is its perspective, intersecting V′E′ at v, the desired perspective of the vertex.

3°. *The perspectives of the extreme visible elements of the cone.*

These will simply be tangents, from v, to the perspectives of the bases, as shown in the figure.

The perspectives just described, can also be found by direct construction from their projections, as follows: Make eE, perpendicular to the ground line through E′, equal to DE′, and E will be the revolved horizontal projection of the point of sight. A visual plane, tangent to the cone, will therefore contain the visual ray EV—E′V′ through the cone's vertex; and its trace


H"
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L'
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M"
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V

on any horizontal plane will contain the intersection of this line with that plane. Thus, EV—E′V′ meets the plane, H′m', of the upper base at K′K, and, accordingly, KN and KM are the traces, on that plane, of two tangent visual planes. Then MV—M′V′ and NV—N′V′—not drawn—are the elements of apparent contour of the cone, and if we find the perspectives of MM′ and NN′ (N′ is near T′, and not lettered) they will be the upper ends, denoted by M″ and t, of the perspectives of these elements.

Problem XLIII.

To find the perspective of a tower.

Let the method of *two planes*, with the perspective plane translated *forward*, be adopted. The perspectives of the points of the tower, Pl. IV., Fig. 33, will be found by various methods from Table III. In the figure above the ground line LL, the outer square is the plan of the base of the tower; the circle, that of its upper and cylindrical member; the dotted square, that of the base of its middle portion; and the octagon, that of the octagonal middle portion. These last parts are merged in one by the triangular uniting surfaces, as fdg, more clearly indicated in the perspective at FGD.

All the points of the figure being easily treated in sets, contained in successive horizontal planes, the vertical traces of these planes, showing the heights of the points in them, are all that is needed as an equivalent for a complete vertical projection of the tower. The plane, A′B′, of the top of the base is taken as that of the horizon. EE″, the point of sight, is taken in a plane perpendicular to the ground line and through the centre of the plan. D′, near v, in Fig. 32, is the vanishing point of diagonals, E″D′ being equal to the distance of E from LL. Or draw ED to D, on LL produced, and project D at D′. Likewise find V′, the vanishing point of all parallels to ak.

The rest of the construction is left for the student to make. Some points of it are found by diagonals and perpendiculars; some, by lines of the object, combined with either diagonals *or* perpendiculars; and some by vertical visual planes, and either one of these auxiliary lines.

EXAMPLE 1.—*Find the perspective of a cone, with a vertical axis, and standing on the horizontal plane; and with the eye in the perpendicular through its vertex.*

Ex. 2.—*Find the perspective of a cone whose axis is parallel to the ground line, by the method of two planes.*

Ex. 3.—*Do. by three planes.*

Ex. 4.—*Find the perspective of the tower by the method of three planes.*

PROBLEM XLIV.

To find the perspective of a cube, whose edges are oblique to both planes of projection.

This problem, taken, with some alterations in the given conditions, and the methods of construction, from LEROY, may serve as a final example at present of the use of vanishing lines of planes and vanishing points of lines.

1°. *To construct the given conditions.*

Let the cube, Pl. XV., Fig. 105, be given by the horizontal trace, and angle of declivity, of the plane containing one of its faces, and by the position of that face in its plane.

Let PQ be the horizontal trace of the plane of the upper face of the cube, and let $C''Pb''=\alpha$ be its angle of declivity, contained in a plane, RP, perpendicular to the trace PQ. Then PC'' is the trace of the plane PQ upon the plane PR, taken as an auxiliary vertical plane of projection, and revolved about its horizontal trace PR, till it coincides with the horizontal plane. PC'' is thus also the auxiliary projection of the top face of the cube.

Now conceive the plane PQ to be revolved about the trace PQ into the horizontal plane, and then let A, in this trace, be one corner of the cube, and Ab and Ad two adjacent edges of the face contained in the plane PQ. The points b and d are projected at b'' and d'', and return in arcs, $b\text{B}-b''\text{B}''$, and $d\text{D}-d''\text{D}''$, perpendicular to PQ, to their primitive positions B,B$''$ and D,D$''$. Then BC, parallel to AD, and DC, parallel to AB, will complete the horizontal projection of the top of the cube.

One of the remaining edges will be sufficient. Then draw

B″F″=AB, since the lateral edges, being perpendicular to the plane PQ, are parallel to the plane PR, and hence are projected upon it in their full size. Also, for the same reason, BF, the horizontal projection of the lateral edge through B, is parallel to the auxiliary ground line RP.

2°. *To find the vanishing lines and points of the cube.*

Now let EE′ be the point of sight, at the distance Q′*q* below the horizontal plane, as RQ is the first, and SQ′ the translated position of the ground line of the perspective plane. This being settled, we next find the vanishing lines of the three planes containing the faces whose common point is A. Thus, E*f*, parallel to RP, is the trace of a vertical visual plane containing the angle of declivity of the given plane PQ. Then revolve this plane about its vertical trace *f*J, and E,E′ will appear at E″,E‴. Then make the angle JE‴E′=C″P*b*″, and E‴J will be the revolved position of a visual ray parallel to the declivity of the plane PQ. Hence this ray pierces the perspective plane at J, where it intersects the trace *f*J of the vertical visual plane containing it; and J is, therefore, one point of the vanishing line of the plane, PQ, of the top of the cube. But (Theor. IV.) this line is parallel to the vertical trace of that plane, which trace must therefore next be found.

The principal vertical trace of RP, is RP′, perpendicular to the principal ground line, RQ, and when this plane is revolved 90° into the horizontal plane, its vertical trace, RP′, takes the position RP″, perpendicular to RP, and it intersects the similarly revolved auxiliary vertical trace C″P at P″, the revolved position of the intersection of C″P with the principal vertical plane. Now, as C″P″ is the intersection of the given plane PQ and the perpendicular plane RP, the point P″ is a revolved point of the vertical trace of PQ. Hence make RP′=RP″ and P′Q, in the lower part of the vertical plane, is the vertical trace of the plane, PQ, of the top of the cube; and, finally, *e*′J*c*′, parallel to P′Q, and through J, is the vanishing line of this same plane PQP′.

The vanishing points of all lines in the plane PQP′, will be in the vanishing line, *e*′*c*′, of that plane. Thus, E*e* is the horizontal projection of the visual ray, parallel to the edge AB of the cube, and which pierces the perspective plane in *e*′*c*′, and also in the vertical line *ee*′, that is at *e*′. Likewise, the visual ray, E*c*, parallel to AD, pierces the perspective plane at *c*′.

Therefore e' is the vanishing point of all parallels to the edge AB, and e' is that of all parallels to AD. But as these lines are edges of the cube, e' and c' are points in the vanishing lines of the lateral faces bounded by these edges, and the visual ray, Ef, being parallel to BF, and hence perpendicular to PQ, pierces the perspective plane in the vertical line fJ. This ray is, in space, perpendicular to the plane PQP′, hence E‴f' perpendicular to E‴J, is its position, after revolution, as in case of E‴J, about the vertical axis fJ. Hence, it pierces the perspective plane at f', which is, therefore, the vanishing point of BF and of all parallels to it. It is consequently also a point in the vanishing lines of the planes containing lateral faces bounded by the edges AB and AD. Hence $e'f'$ and $c'f'$ are these vanishing lines.

3°. *The perspective of the upper base of the cube.*

The foregoing preliminaries now permit the immediate construction of the required perspective, as follows: The edge, BA, pierces the perspective plane in the trace P′Q, at g', whose translated position is g'', giving $g''e'$ as the indefinite perspective of AB. But the visual ray, AaE, pierces the perspective plane in $g''e'$, and in the vertical trace, aA′, of its projecting plane upon the horizontal plane; that is, at A′, which is therefore the perspective of A. Then A′e', A′c', and A′f', are, at once, the indefinite perspectives of the three edges which meet at A. To limit them, let us first draw the face diagonal, AC, and the parallel visual ray Ek. As this ray is in the plane PQ, its vanishing point, K, will be in $e'c'$, the vanishing line of that plane, and at its intersection with kK, the projecting line of the point k. Hence A′K is the indefinite perspective of AC, and it is limited at C′, where the visual ray EC pierces the perspective plane at h,C′, giving C′ as the perspective of C. Then e'C′ will limit A′c' at D′; and cC′ will limit A′e' at B′, which completes the perspective of the upper base.

4°. *To complete the perspective of the cube.*

EL, parallel to AF, is the horizontal trace of a vertical visual plane parallel to the diagonal, AF, of a lateral face of the cube. Hence the vanishing point of this diagonal is L′, the intersection of LL′, perpendicular to the ground line, and the vanishing line, $e'f'$, of the face ABF. Likewise M′ is the vanishing point of the diagonal, through A, in the left hand front face of the cube. Therefore, A′L′ limits B′f' at F′, the perspective of

F′F″; likewise, e'F′ limits A′f' at G′; either G′K, or F′c', or D′L′, limits C′f' at I′, and either A′M′ (not shown), or e'I′, limits D′f' at H′.

Thus, error at each point is checked by having several constructions for the same point. Indeed, the initial point A′ is verified by being on the line Q′V, the perspective of the horizontal trace PQ, which pierces the perspective plane at Q, whose translated position is Q′, and whose vanishing point is at V (where the horizon intersects $e'c'$).

EXAMPLE.—*Find the perspective of a regular tetraedron, one of whose bases is in a plane parallel to the ground line but oblique to the planes of projection; and find the vanishing lines of its faces, and the vanishing points of its edges.*

CHAPTER IV.

SPECIAL PRINCIPLES AND OPERATIONS.

SECTION I.

Special Theorems.

THEOREM VII.

All equal lines, in a plane parallel to the perspective plane, have equal perspectives.

Let Q, Pl. V., Fig. 35, be a plane containing the given lines A, B, and C, having any position in that plane; and let P be the perspective plane, parallel to Q. Connect the given lines in any convenient manner, as shown by the dotted lines in the plane Q. The base of a pyramid whose vertex is E, the eye, will thus be formed.

Now the section of this pyramid, made by the parallel plane P, will be similar to the base, and as A, B, and C are equal, a, b, and c will be equal also. But the latter are the perspectives of the former, which proves the theorem.

Fig. 34 merely shows in projection the case where $a'a''$ is the common vertical projection of the parallel lines a, b, and c, to which the perspective plane, PQP′, is parallel.

Remark.—A, B, and C *are* equal but would not *appear* so, from E, being at unequal distances from it. The same is true of a, b, and c. Disregard of this very elementary principle is the occasion of the most amusing popular fallacies about perspective.

THEOREM VIII.

The perspectives of a line, parallel to the perspective plane, will be equal, wherever the eye is placed, in a plane, parallel to the perspective plane.

This theorem, which is the reciprocal of the preceding, where the eye was fixed, is sufficiently proved by reference to Pl. VII.,

Fig. 60, where AB is the line, E and E′ two different positions of the eye in a plane whose trace is EE′, parallel to AB, and *ab′* is the position of the perspective plane, containing *ab* and *a′b′*, the perspectives of AB, seen from E and E′ respectively, as shown by the visual rays AE, etc.

Now the triangles EAB and E′AB, having a common base and equal altitudes, are equal; and *ab′* being parallel to EE′, the altitudes of E*ab* and E′*a′b′* are equal. Also, *ab′* being parallel to AB, E*ab* is similar to EAB, and E′*a′b′* to E′AB. Hence E*ab* and E′*a′b′*, having equal altitudes, and being similar to equal triangles, are themselves equal, on equal bases *ab* and *a′b′*. But these last lines are the perspectives of AB, hence the theorem is established. See, also, Fig. 61, where, likewise, *ab* and *a′b′* are the equal perspectives of AB—A′B′.

THEOREM IX.

The perspectives of all lines, in a visual plane parallel to the ground line, will, themselves, be parallel to the ground line.

An indefinite number of visual planes can be passed through a parallel to the ground line, drawn through the eye, each of which will intersect the perspective plane in a trace, parallel to the ground line. But, by the definition of perspective, this trace is the perspective of any, or all, lines in the visual plane, which proves the truth of the theorem. For illustration, see Pl XI., Fig. 80, where *ad—a′d′* is a line parallel to the ground line GG′. Now E*a*—E′*a′* is the visual ray from a point, *aa′*, of this line, and it pierces the perspective plane at *b′*, the perspective of *aa′*, and gives PQ parallel to GG′, through *b′*, for the perspective of *ad—a′d′*. But join any point, as *cc′*, of a visual ray from any point, *aa′*, of the given line, with any point as *dd′* on that line. The line *ed—c′d′* will then not be parallel to the ground line, but, being in the same visual plane with *ad—a′d′*, its perspective, as appears by construction, is evidently the same parallel, PQ, as before.

THEOREM X.

If any straight line intersects the perpendicular visual ray, its perspective and its vertical projection will coincide.

The perspective of a straight line is the trace of the visual plane, passed through it, upon the perspective plane. In the present case, this visual plane is determined by the given line and the perpendicular visual ray which it intersects. As the latter is perpendicular to the perspective plane, the plane containing it must be so also. Hence it is merely the projecting plane of the given line, upon the perspective plane; hence the perspective and the vertical projection of that line are identical; the perspective and vertical planes being also understood to coincide.

THEOREM XI.

If a system of lines meet in any point of the plane, through the eye, parallel to the perspective plane, the perspectives of these lines will be parallel.

Conceive of A, B, and C, as three lines meeting at any point, M, in the given plane, P, parallel to the perspective plane, and let E be the point of sight, in this parallel plane. Then the visual plane of each of the given lines, A, B, and C, will contain both M and E; that is, EM is the common trace on the parallel plane, of the three visual planes. But the perspective plane, being parallel to the plane P, the traces, upon it, of these visual planes, *which traces are the perspectives of the given lines*, are parallel to EM, and hence to each other, as stated.

See also Fig. 12, p. 99, where eE'' is the common trace, on the plane AQ, of the visual planes through UE″ and YE″, which meet in the visual plane AQ, parallel to the perspective plane GLP; and where pu' and qy', parallel to eE'', are the parallel perspectives of UE″ and YE″.

THEOREM XII.

From the trace of a line, on the perspective plane, to its vanishing point, is the perspective of an infinite extent of the line, from the perspective plane, and forward from the eye.

See Fig. 6, where AGL is the horizontal plane; VGL, the

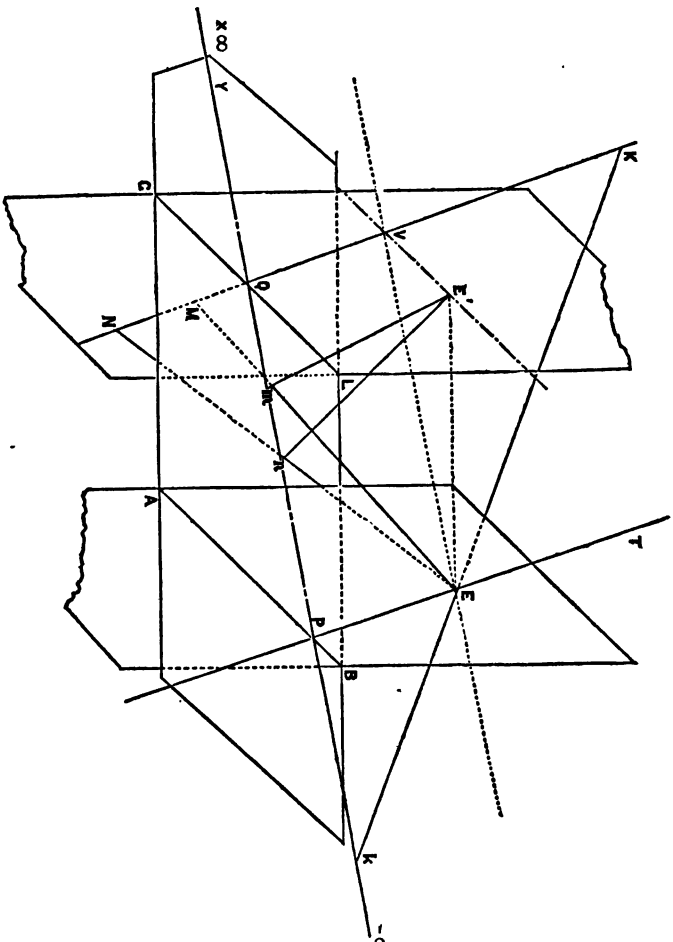


FIG. 6.

perspective plane; E, the eye; and PY, a line; in this case, in the horizontal plane.

Q is the trace of this line, and EV, being the visual ray, parallel to it, V is its vanishing point (32). Hence QV is the perspective of QY, extended to an infinite distance from Q, as stated.

THEOREM XIII.

The perspective of that portion of a straight line, intercepted between the perspective plane and the parallel visual plane, extends from its trace, to an infinite distance, opposite to its vanishing point.

In Fig. 6, again, let AEB be the visual plane, parallel to the perspective plane. Then PQ is the portion of the given line, *k*Y, between these planes.

Now, take any point, as *m*, on this line, and between the two planes, and E*m*, the visual ray through it, evidently pierces the perspective plane in a point, M, on the opposite side of the trace, Q, from the vanishing point, V. Likewise, N is the perspective of *n*; and, finally, the visual ray, EP, being parallel to the perspective plane, would meet it only at an infinite distance. Hence, as stated, the perspective of QP extends from Q to infinity in the direction QN.

THEOREM XIV.

The perspective of that portion of a straight line, which is behind the visual plane, parallel to the perspective plane, extends from its vanishing point to infinity, and opposite to its trace.

See Fig. 6, once more, where P*k* is the portion mentioned, of Y*k*.

The visual ray, EP, meets the perspective plane at an infinite distance, in the direction ET, as well as in the direction EP. Also, the visual ray EV, meets PQ at infinity backwards as well as forwards, or in the direction QY. Hence V is the perspective of the point infinitely far *behind* the plane EAB, as well as of the point infinitely far *in front* of it. And, finally, taking any point, *k*, at pleasure, between P and $-\infty$, its perspective, *k*, is evidently on the opposite side of V, from the trace Q.

Hence, as stated, the perspective of P to $-\infty$ is VK, extended to infinity.

Remark.—The above figure peculiarly well illustrates the use of such exact, yet pictorial constructions, in demonstrating

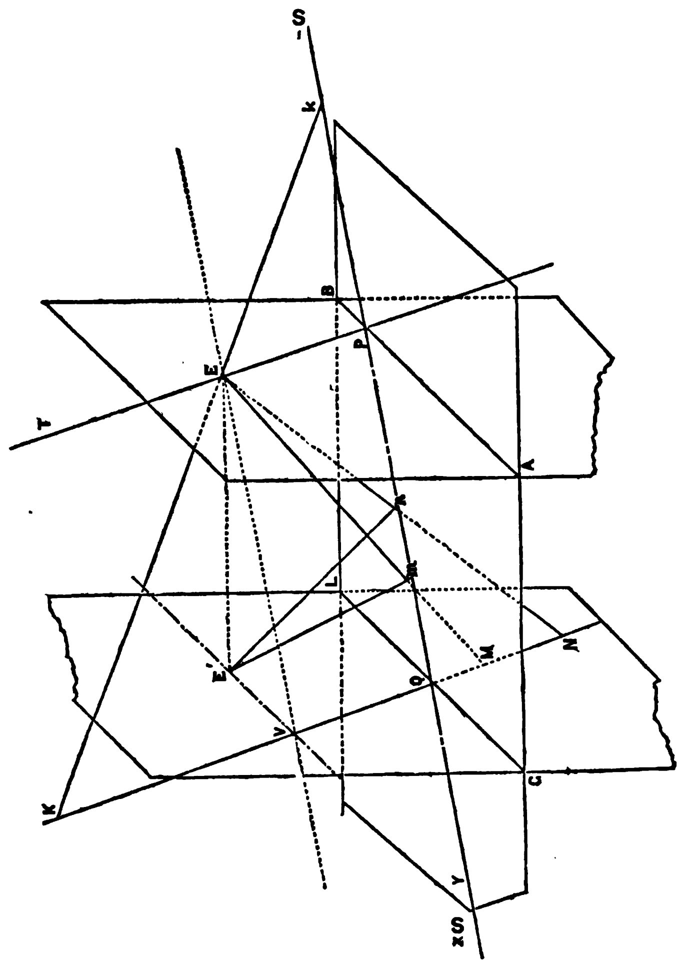


FIG. 6.

those principles, which would be especially perplexing, as illustrated by ordinary diagrams.

Fig. 6, and all similar figures, are oblique projections, of the kind called "cabinet projection" in my Elementary Projection Drawing; and called, also, "military" or "mechanical" perspective.

SECTION II.

Of perspective, and original Lines and Angles, and their Division.

PROBLEM XLV.

Having the perspective of a horizontal straight line, of known height above the horizontal plane, to find its trace; its vanishing point, its original, and the perspective of a line, making any given angle with it at a given point.

1°. Let A′B′, Pl. V., Fig. 36, be the given perspective, and *eh*, the height of its original. Then the trace of the latter is in the parallel, through *h*, to the ground line, and in B′A′ produced, hence, at T′T, since GL is the original ground line.

2°. Let *d*′V be the horizon, then by producing A′B′ to V, we find V for the vanishing point.

3°. EE′ being the point of sight, project V horizontally at *v*, and *v*E (32) is a visual ray, parallel to the original of A′B′. Then a line from T, parallel to E*v*, will be the indefinite original line. To find its limits, draw a perspective perpendicular, or diagonal, as the diagonal *d*′*g*′, and its original, *g*A, which makes an angle of 45° with T*g*, will meet TA at A, the original of the perspective point A′. Let the original of B′ be found by a perpendicular.

4°. Any other horizontal line through E, besides E*v*, will be the visual ray which will determine the vanishing point of a line parallel to it, and intersecting AB, at an angle equal to the one thus having E for its vertex. Thus, first, if this is to be a right angle, draw E*p* perpendicular to E*v*, and project *p* at P, the vanishing point of all perpendiculars to AB. Hence if C′ be the given point on A′B′, the line PC′ will be the required perspective of a perpendicular to AB. Or, second, if *f*E*v* be the desired angle, F is the vanishing point of all lines making that angle with AB, and FC′ is the perspective of a line through C′ as required. Finally, any line, as VD, through V, is the perspective of a parallel to AB.

Remark.—The method of dividing the perspective of an angle in parts whose originals shall be equal, is now obvious.

PROBLEM XLVI.

Having the perspective of a perpendicular, to find the perspective of a point on it, a foot behind its trace; a second point, any number of feet back of the first; to divide its perspective into n *parts whose originals shall be equal; and to find the true length of those originals.*

1°. Let C, Pl. V., Fig. 37, be the centre of the picture, and A′C the perspective of a perpendicular, whose original is obviously BA, when A′ is given as its trace. Then let A*a* = 1 ft., and *ab*, being a diagonal, A*b*, which = A*a*, also = A′*b*′, by projecting *b* at *b*′, on A′*c*′, drawn parallel to the ground line A*c*. Therefore if D is the vanishing point of diagonals, *b*′D will give *a*′, as the perspective of *a*.

2°. Observing that A′*b*=A*b*=A*a*, make *b*′*c*′ equal any number *n*, as 3, feet, and the perspective diagonal *c*′D cuts off *a*′B′, the perspective of a distance three feet back of *a*.

3°. All parallels dividing *a*B will have a common vanishing point, and will include equal spaces from *b*, on A*b* produced; hence from any vanishing point, V, draw V*a*′ and VB′, and produce them to A*c*′, at *h* and *g*; divide *hg* equally, as at *d* and *e*, and *d*V and *e*V will give the perspectives *a*′*d*′, etc., on A′B′, of equal parts of AB.

4°. Observing again that A′*b*′=A*a*, draw perspective diagonals from D, through *d*′, etc., and they will give on A′*c*′ equal spaces, equal to the originals of A′*a*′.

Remarks.—a. As a single projection cannot determine a magnitude in space, neither can a single perspective. For a line, L, in the perspective plane, is the perspective of *any* line in a visual surface through L. But if we have two perspectives of the same object, and from two points of sight, the intersection of the two visual surfaces, containing those perspectives, will be their fixed original.

b. Distances in perspective, whose originals are equal, are briefly said to be perspectively equal.

c. Observing that *ab* makes equal angles with AB and A*c*, as an essential feature of the construction, it is evident that this problem will include the case of any line having a vanishing point in the horizon, by using the method of Prob. XXV.,

where, instead of diagonals, chords of arcs having the trace, as T, for a centre are the lines of construction.

Problem XLVII.

Having the perspective of an oblique line, in a given plane, parallel to the perspective plane: to find its foot; its length; a given distance on it; and to divide it equally.

Let A′B′, Pl. V., Fig. 40, be the line, CV the horizon, and AB″ the ground line.

1°. Suppose the line to be one foot behind the perspective plane. Then CP is a perspective perpendicular, and, making PQ = one foot, and drawing the perspective diagonal QD, we find T, one point of the perspective of a line, parallel to the ground line, and one foot from it, in the horizontal plane. T*b* being thus the horizontal trace of a plane, through A′B′, and parallel to the perspective plane, A′, its intersection with A′B′, is the required foot of that line.

2°. As before, any parallel horizontals will intercept on the perspective plane the true length of A′B′. Hence draw CA through A′, and AB, parallel to A′B′, and limited by CB′, and AB will be the true length of A′B′. Otherwise: Revolve A′B′ about its foot A′, into the trace A′*b* and it will, being still in the perspective plane, preserve its length unchanged. Then, lines through A and *b*, from any point as V on the horizon, will be the perspectives of parallels, in the horizontal plane, and will therefore intercept on the ground line a distance, A″B″ equal to AB, the true length of A′B′.

3°. To find the perspective of a point, one foot for example, from A′. Make A*a* = one foot, and draw *a*C, which gives A′*a*, the required perspective; or, make A″*m* = one foot, and draw *m*V, giving A′*n*, the revolved perspective of A*a*=A″*m*; then revolve A′*n*, about A′ as a centre into A′B′, giving A′*a*′ as before.

4°. To divide A′B′ into parts perspectively equal, divide A′B′ itself directly into equal parts, since it is parallel to the perspective plane. It may likewise be divided into parts of any given relative size.

PROBLEM XLVIII.

Having given a plane, perpendicular to the ground line, by its perspective horizontal trace, and vanishing line, to construct the perspective of a line in it, making a given angle with the horizontal plane, at a given point in the given trace.

Let AC, Pl. V., Fig. 42, be the perspective horizontal trace of the given plane, which is supposed to be vertical. As C is the centre of the picture, the plane is perpendicular to the ground line; and BCV, perpendicular to that line, is its vanishing line. Now revolve the point of sight into the horizon, at C″, since C″C is supposed to be the true distance of the eye in front of C. The visual ray, which determines the vanishing point of the required line, is parallel to it, hence if C″B be drawn, so as to make the given angle with C″C, it will be the revolved position of that ray, and will meet CB in B, the vanishing point sought. Then, supposing A to be the given point of the horizontal trace AC, AB will be the perspective of the line required.

Remark.—A perpendicular to LL, at T, will be the original horizontal trace whose perspective is AC.

EXAMPLE.—*Solve this problem with AC to the right of BV, and with B below the horizon.*

PROBLEM XLIX.

Having the perspective of a line in a given plane, perpendicular to the ground line; to find its traces and vanishing point; the perspective of a given distance on it; and of any proposed parts of it.

Let *a′b′*, Pl. V., Fig. 42, be the given line, AC, the perspective horizontal trace of a vertical plane through it, and BCV the vanishing line of that plane. Then—

1°. A, the intersection of *b′a′* and CA, is the *perspective* of the horizontal trace of the line, and T′, the intersection of the vertical trace of the plane with *b′a′*, is its vertical trace. Draw T′Q′ (not shown) parallel to BC″, and meeting LL at Q′. Then, with T as a centre, revolve Q′ to Q on T′T produced, and Q will be the original horizontal trace of BA. Also B, the intersection of *a′b′* with CB, is the vanishing point of *a′b′*.

2°. To find the perspective of one foot, for example from a', conceive the line Ab' to be revolved about A as a centre into the trace AC. All the revolving points of Ab' will thus describe arcs, the chords of which will be parallel. But C″C, and C″B, are the revolved positions of visual rays, respectively parallel to AC and AB; hence BF is the revolved position of a parallel to one of the chords just described; and C″V, parallel to BF, is the revolved position of a visual ray parallel to these chords. This ray evidently pierces the perspective plane at V, it being in the plane ACB, and V is therefore the vanishing point of these chords. Va' now represents the chord of the arc described by a', and d is the perspective revolved position of this point. As AC is, in space, a perpendicular, diagonals will intercept equal spaces on it and on the ground line, hence draw the perspective diagonal C″p, through d, make pq to represent one foot, draw the diagonal qC″, meeting AC in c, and draw Vcc', giving $a'c'$ the perspective of one foot on $a'b'$, measured from a'.

3°. Let the portion $a'b'$ be divided equally. The method of Prob. XLVI. might be applied to de, the perspective revolved position of $a'b'$, and then the points on $a'b'$ could be found as just shown. But as lines and their horizontal projections are always divided proportionally, it is sufficient to find ab, the perspective horizontal projection of $a'b'$, by drawing the verticals $a'a$ and $b'b$, and to apply to ab the construction of Prob. XLVI., and, finally, to erect verticals, at the points thus found on ab, which will divide $a'b'$ as required.

Problem L.

Having a plane, which is perpendicular only to the horizontal plane, to find the perspective of a line in it, and making a given angle with its horizontal trace, at a given point.

Let AV, Pl. V., Fig. 43, be the perspective of the given horizontal trace, CC′ being the point of sight. Cv—C′V is then the visual ray which determines the given vanishing point, V, and as the given plane is vertical, vVB′, perpendicular to the horizon, is its vanishing line. Now revolve CC′ about B′v to GG′, and draw G′B′ making the angle B′G′V equal to the given angle, and G′B′will be the revolved position of the visual

ray, parallel to the required line in the given plane. Hence B′, in the vanishing line of that plane, is the vanishing point of the line; and, if A be its given horizontal trace, AB′ will be the required perspective of the line.

PROBLEM LI.

Having given the perspective of a line, in a given plane; to find its traces and vanishing point, a required perspective length upon it, and to divide it in any required manner.

Let *a′b′*, Pl. V., Fig. 43, be the given line in the plane whose perspective horizontal trace is AV, and whose vanishing line is *v*VB′.

1°. Produce *a′b′* to A, on VA, which is its perspective horizontal trace. See also Prob. XLIX. T″, on the vertical line T′T″, is its vertical trace. Also B′, the intersection of *a′b′*, produced, with VB′, is its vanishing point.

2°. As in the last problem, lines from points on AB′, and parallel to the position of the chord B′F in space—the arc BF having G′ for its centre—will be the chords of arcs described by points of AB′ in revolving about A as a centre, and into AV. The vanishing point, P (below the border), of these chords, is the intersection of the revolved visual ray G′P, parallel to B′F, with B′V produced. Hence draw *a′*P, and *a″* will be the perspective revolved position of *a′*. Then apply on AV the method of Prob. XLVI., only remembering that, as AV is not a perpendicular, lines, parallel to GC, and not diagonals, will intercept equal spaces on TB (parallel to C*v*, and the original of AV), and on the ground line. G′ is the vanishing point of parallels to CG, hence draw G′*a″*, produce it to *p*, make *pq* equal to the given distance, as one foot, and note the point *c*, draw *cc′* from P, for the perspective chord of the arc in which *c* revolves back to *c′*; and *c′* will be the perspective of a point one foot from *a′*.

3°. To divide *a′b′* in any desired manner, for example, equally. Here, as in previous cases, *vertical parallels* will divide *a′b′*, and its *perspective horizontal projection*, *ab*, similarly. Also *chords*, vanishing at P, will divide *a′b′*, and its *perspective revolved position*, of which *a″* is one extremity, similarly. Likewise *any* horizontal parallels will divide *ab*, or

$a''b''$, and the portion of the ground line included between the parallels, proportionally. Hence from any point, M, on the horizon, draw Ma and Mb, through to the ground line, at m and h, divide mh equally, as at n and r, or in any desired manner, draw nM and rM, giving d and e, from which draw verticals dd' and ee', which will divide $a'b'$ as required, at d' and e'.

Remark.—Lines from G′, through a'' and b'', would intercept, on the ground line, the true length of $a'b'$.

SECTION III.

Of the Methods by Reduced Distances, and Transversals.

Problem LII.

To construct the perspective of a point, by reduced distances; in the case of a distant vanishing point.

In the practical applications of perspective, it usually happens, that the eye is taken, in order to produce the best pictorial effect, so far from the perspective plane, that the vanishing points of diagonals and other oblique lines are very far from the centre of the picture. In these cases, one way of proceeding is as follows, *by reduced distances.*

Let A, Pl. V., Fig. 41, be the vertical trace of a perpendicular, whose perspective is AC′, and suppose the perspective of a point, four feet from A, on AC, to be required. Draw AP, parallel to the horizon C′D, and make AP equal to four feet. Then, D, being the vanishing point of diagonals, PD would cut off from A the perspective, Am, of four feet. But if D were inconveniently far from C′, we could make $C'd = \frac{1}{2}$ C′D and A$p = \frac{1}{2}$ AP, and pd would pass through the same required point m.

Again: let BV be the perspective of *any* horizontal line, not a perpendicular. Revolve the horizontal visual plane—taking the method of Prob. XXV.—about C′D as an axis, into the perspective plane, when C′C will be equal to C′D. Then CV will be the visual ray parallel to the original of BV, and revolving C about V as a centre to C″, CC″ will be the visual ray, parallel to the chords which would intercept equal distances on the original of BV, and on the ground line; or, in

general, on the horizontal line through the vertical trace of the original line. Hence C″ is the vanishing point of these chords, and QC″ is the perspective of one of them, intersecting BV at M, so that MB is the perspective of a distance equal to BQ. But, now, if BQ and VC″ are inconveniently great, make $Bq = \frac{1}{2}$ BQ, and $Vc'' = \frac{1}{2}$ VC″, and qc'' will pass through the same point M, on BV, that was before found.

PROBLEM LIII.

To construct the perspective of a given angle, by the method of reduced distances; one side of the angle being given; or, to divide a given perspective angle.

Let AB, Pl. VI., Fig. 47, be the given perspective line, VH, the horizon, and CC′, the reduced distance of the point of sight; after revolving the horizontal visual plane into the plane of the paper, and making C′C equal to half, or any simple part, of the real distance of the eye from the perspective plane. AC′ is the perspective of a perpendicular from A; but the size of the perspective of a fixed line at a fixed distance varies directly as the distance of the perspective plane from the eye. Hence, if C′C is, as here supposed, one-half of its full size, C′D should be one-half of C′A, and DV, parallel to AB, will be the reduced perspective of AB, and V will be its vanishing point. Then VC will be the new visual ray, parallel to the original of AB. Hence, make V*Cm* equal to the given angle, and draw *m*D, and AM, parallel to *m*D, will be the perspective of the other side of the required perspective angle BAM.

If, now, the angle BAM were to be divided in any proposed manner, we should divide its original size, V*Cm*, in the same manner, as by the line *Cn*, for example; then draw *n*D, and its parallel, NA, will divide BAM perspectively, as required.

The whole principle of the above construction is, that perspectives of the same object on parallel perspective planes, are similar figures, being parallel sections of the same visual cone; and that their size depends on the distance of the perspective plane from the eye.

PROBLEM LIV.

To find the perspectives of horizontal lines, nearly parallel to the perspective plane, by reduced distances, having given one such perspective.

Let the required line pass through P, Pl. VI., Fig. 48, and be parallel, in space, to AB, and let HH′ be the horizon.

Draw any two perspective lines AH and PH, and from any point on either, as a, draw ap, parallel to AP. Then draw aH′, parallel to AB, and join p with the reduced vanishing point, H′. Then PO, parallel to pH′, will be the perspective line required. The line joining b and q will be parallel to PA, if the construction be exact. In other words, ab and pq, produced, will be parallel to AB and PO, respectively, and will meet at the same point, H′.

75. The demonstration of this construction is a beautiful example of the application of the principles of solid geometry to the properties of plane figures, considered as projections of certain solids. Thus V—APQ (V being the intersection of AB and PO, on the horizon HH′) may be viewed as the projection of a triangular pyramid, whose vertex is Q, and which is truncated, at bqH′, by a plane parallel to its face APV. So that, if bH′, qH′, and QH′ all meet at one point H′, the parallel set of lines ABV, POV, and QV, must all have a common point, V, also.

Or, calling APQ and APH two plane sections, having a common edge AP in the face APV, the lines ap and bq, being cut from these sections by the plane H′bq, parallel to that face, must both be parallel to AP, and hence to each other, as was stated.

76. The construction, just made, can also be proved by plane geometry, thus: In the similar triangles H′Qb and VQA.

$$H'Q : Qb :: VQ : QA.$$

Likewise, $$H'Q : Qq :: VQ : QP.$$

$$\therefore \quad Qb : QA :: Qq : QP.$$

Hence bq is parallel to AP. But ap is, by construction, parallel to AP, hence ap and bq are parallel, when H′ and V are, as constructed, each the point of concourse of three lines as shown.

Problem LV.

To find the perspectives of horizontal lines, nearly parallel to the perspective plane, by means of transversals.

Case 1°.—As before, Pl. VI., Fig. 48, let AB be a given line, and P, a given point. Draw any transversal, PA, and, at a convenient distance, OB, parallel to it. Make FO a fourth proportional to AE, EP, and BF, and PO will evidently meet the horizon, HH′, at the same point with AB, and will, therefore, be perspectively parallel to it.

Case 2°.—Let CD, Pl. VI., Fig. 49, be the horizon, QG the given line, and P the given point, between QG and CD. Through P, draw any two transversals, QPB and OPA. Draw QA and OB, and note their intersection E. Join E with any point, G, of the given line, and draw the diagonals OF and GB, in the quadrilateral, OGFB. Their intersection, H, will be a second point of the required line, through P, and parallel to QG, in space. To prove this, it is only necessary to consider that QABO and OBFG may be regarded as the perspectives of adjacent parallelograms, having the original of OB for a common side, and each having a side in QG, and an opposite side in the visual plane through CD. The remaining sides would then be parallel, and E represents their vanishing point. Now the centres of these rectangles would be the intersections of their respective diagonals, and a line joining these centres would evidently be parallel to the continuous sides in QG. PH is such a line, and is therefore perspectively parallel to QQ.

By reading *ph*, etc., for the above corresponding large letters, we have the case where P is midway between CD and QG.

Case 3°.—Let P, Pl. VI., Fig. 50, be exterior to the horizon CD, and given line RS. Draw PA and PB, to any points A and B, on CD; and the diagonals, AS and BR, of the quadrilateral ABEG, meeting at *e*. Then draw P*e*M, meeting RS at E, and next MRN and MSH, limited by BEN and AEH; and NPH will be the required perspective parallel to HS.

The principle of this construction is nearly the same as in the last case, only the application is not quite so simple. As PM passes through the perspective centre, *e*, of the perspective parallelogram ARSB, we have AREM, and BSEM, equal paral-

lelograms in space; and hence MN and BN, MH and AH, AR and BP, and RS and NH are, in space, *four pairs of parallels, in the same plane,* hence they all have their vanishing points, N,P, etc., in the same line (42—4°).

Case 4°.—*The simplest construction* of a line, perspectively parallel to a given perspective, is shown in Pl. VI., Fig. 51, where AD is the horizon, and either of the other lines, as B*b*, may be a given perspective, and C, accordingly, one point of the required perspective parallel. Join C with A and B, taken at pleasure, and draw AB. Then make *ac*, anywhere, parallel to AC, and *ab* and *bc* parallel to AB and BC. Then C*c*, B*b*, and AD will all concur at a common vanishing point.

This construction is also explained by the principles of projections, regarding V—ABC as a pyramid, in which *abc* is a section, parallel to the base.

In each of the last two problems, CD may be regarded as the vanishing line of *any* plane, to which the originals of the perspective line, as B*b* and C*c*, Fig. 51, are parallel, those lines having any position in space.

Example.—*Having given, by their projections, two parallel lines, oblique to both planes of projection, but nearly parallel to the perspective plane, find, first, the vanishing line, which will contain their vanishing points* (42—4°), *and, second, their perspectives, by the last two problems; that is, without finding their vanishing point.*

SECTION IV.

Of the Method by Scales; and the Adjustment of the Object—the Eye, and the Picture.

77. This Method, founded on the two preceding sections, and as perfected by Adhemar, is of too great practical value not to receive separate notice. Partial illustrations of it have been made in Section II. of this Chapter, as will be evident on becoming familiar with the principles and operations of the present section. But the origin and value of it will here be more clearly explained.

In all diagrams, designed to illustrate the principles of perspective, the point of sight is taken very near to the perspective plane, in order to avoid having inconveniently distant vanishing points. Hence, when viewed in relation to their

pictorial effect, they are quite offensive to artistic sense, and so tend to repel artists and designers from the study of perspective, as a science, by leading them to think it useless for effecting true constructions of objects, seen at suitable distances for producing the best pictorial effect.

78. It is, however, the object of the *Method of Scales*, in connection with that of *reduced distances*, and *transversals*, given in the last section—

1°. To avoid the use of any points of construction outside the limits of the picture.

2°. To avoid, to the utmost, the need of making complete projections of the object shown, by using known co-ordinate measurements, in place of projections; and yet—

3°. To secure a true, and an agreeable perspective, that is, one due to a suitable distance of the eye both from the *object* and from the *picture*.

79. Of these three things, viz., the *eye*, the *object*, and the *picture*, or perspective plane, any two may be fixed, and the other, variable in position, so that it is necessary in the applications of perspective to industrial practice, to take account of the real and apparent sizes of the perspective figures, separately considered, as well as of those of the original object. The following combinations should therefore be noted.

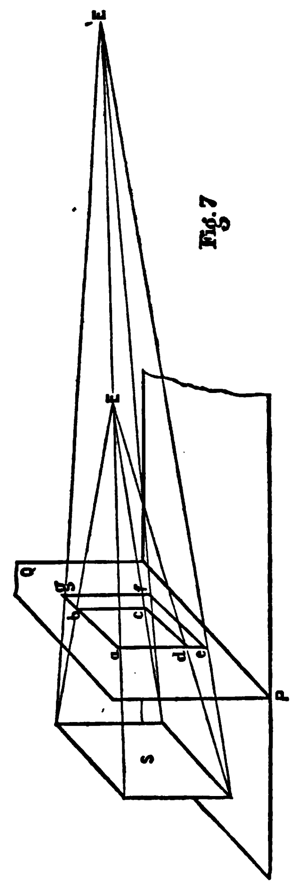


Fig. 7.

THEOREM XV.

The perspective plane, and the object, being fixed; the perspective will really be larger, but will appear smaller, the further the eye is from the object.

For see Fig. 7, where PQ is the perspective plane, at a fixed distance from the original square, S, whose perspective is to be found. When the eye is at E, the

perspective of S is *abcd;* but when it is at E′, the perspective is the larger figure *agfe;* which agrees, so far, with the enunciation.

This fact is, however, also evident from *principle*, as well as by inspection. For the longer the visual cone, ES, the greater must be the section, *agfe*, cut from it by a plane at a fixed distance from its base.

The larger perspective will, however, as stated, have the lesser *apparent* size because, being more distant, it subtends a smaller visual angle, at E′, than at E.

THEOREM XVI.

The picture, and the eye, both being fixed; the further the object is from both, the smaller the perspective figure will both be, and appear.

See Fig. 8, where E, the eye, and PQ, the picture, are fixed, and S and S′ are two parallel positions of the same square.

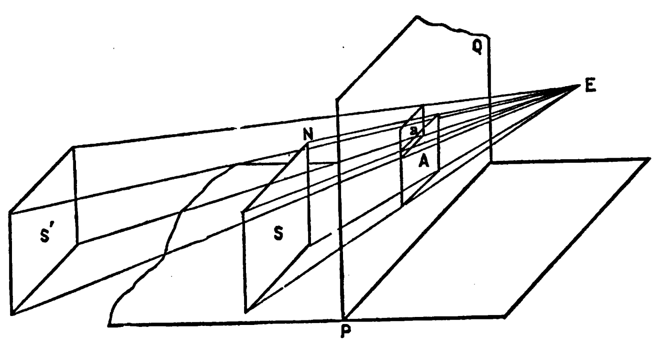


FIG. 8.

Then the figure, A, is the perspective of S, and the lesser figure, *a*, is the perspective of the equal, but more distant square, S′.

The figure, *a*, is *actually* less than A, because the longer the visual cone, having the same base, S, the smaller will be the angle at its vertex, and hence the smaller the section, *a*, cut from it by a plane, PQ, at a fixed distance from its vertex, E. The figure *a* also *appears* less than A, it being really smaller and at the same distance; and this should be so, since S′, being further from the eye than S, it must *appear* smaller.

THEOREM XVII.

The object, and the eye, both being fixed, the nearer the picture is to the eye, the smaller the perspective will really be, though its apparent size will be constant.

This theorem, being more familiar than the other two, needs no separate illustration. The eye, and the object, being fixed, the visual cone, and the visual angle at its vertex, are invariable; hence the different sections of it, made by different positions of the perspective plane, all subtend the same visual angle, and hence will be of *different real*, but *equal apparent* sizes.

If, however, the dimensions of the picture be fixed, the extent of the view, which can be represented upon it, will be greater, the further the picture or perspective plane is from the object.

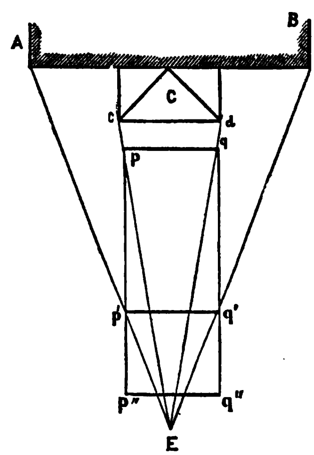


FIG. 9.

Thus, Fig. 9, if AB be a plan of the front of a house, and C a porch, and E the eye, a canvas, or paper, of the length pq, would barely contain the perspective of the porch alone, as seen by the visual rays Ec and Ed. But if the same surface take the position $p'q'$, the *form* of the perspective figure will be the same but smaller, and the whole house can be shown; while, if pq be further moved away from the house, some sur-

roundings of sky and landscape can be admitted into the picture.

80. In the three last theorems, we have spoken of the *sizes*, real or apparent, of perspectives under various conditions. Their *forms*, for a fixed position of a given object, depend only on the distance of the eye from the object, so long as the different positions of the perspective plane are parallel, and the

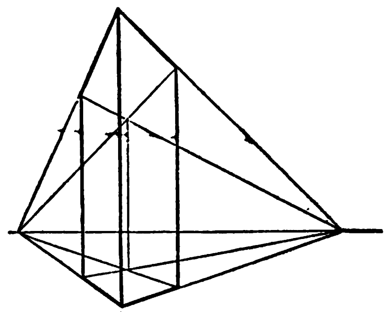


FIG. 10.

object is viewed in the same direction. Thus, Fig. 10 represents the apparently distorted perspective of a square prism, or pillar, seen from a point too near it, while Fig. 11 represents a

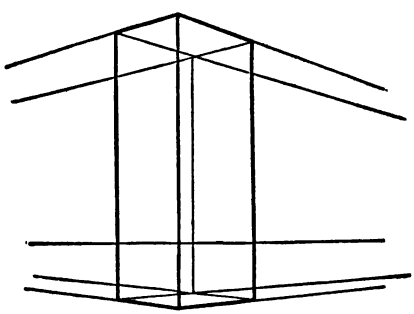


FIG. 11.

view, more agreeably conformed, in the apparent relative direction of its edges, to common experience.

In studying good pictorial effect, therefore, regard must be had to the actual dimensions of the picture; to the distance of the perspective plane from the object, and of the eye from the

object. The latter distance, particularly, should be accommodated to the practical limits of the visual angle, which are at from 30° to 45° apart. Such a perspective as Fig. 10 is in disregard of these limits, and represents the supposed pillar as seen from a point too near for it to be wholly seen at once; that is, to be seen without turning the head.

THE METHOD OF SCALES.

81. Resuming, now, the *method of scales*, let all horizontal distances, parallel to the perspective plane, be called *lengths;* let all those, perpendicular to the same plane, be called *depths;* and let all vertical distances be called *heights.*

We now say that the perspective of any point can be found, when there are given its perpendicular distances, from any plane perpendicular to the ground line, from the perspective plane, and from the horizontal plane. For illustration, see the following problem.

Problem LVI.

To find the perspective of a point in space, by the method of scales.

Pl. V., Fig. 44, is a plan view, only. AB is the ground line, on which the perspective plane is perpendicular to the paper. EE′ is the perpendicular distance of the eye in front of the perspective plane. AN is the horizontal trace of a vertical plane of reference, perpendicular both to the horizontal and the perspective planes. M is the plan of a point, six feet behind the perspective plane, and four feet four inches to the right of the plane AN. Its height is shown in the other figures.

Now, in Figs. 45–46, let A be the lower left hand corner of the picture, let the ground line, AB, be the base of the picture, and the vertical trace, AN′, of the vertical plane of reference, the left hand border of the picture. Let VE′ be the horizon, and E′ the centre of the picture, or vertical projection of the point of sight.

Next, let A be the zero point of three scales; of AB, as a scale of *lengths*, or distance from the plane AN′, or AN, Fig.

44; of AB, again, as a scale of *depths*, really existing on the horizontal trace AN, Fig. 44; and of AN′ as a scale of *heights*. The scale of *depths* can be laid off on AB, because a diagonal from any point, as N, Fig. 44, six feet behind the perspective plane, will cut off the same distance, six feet from A to the right or left, on the ground line AB.

Since AN is perpendicular to the perspective plane, its perspective is AE′.

As the vanishing points of diagonals are at a distance from E′, equal to EE′, Fig. 44, we resort to the method of reduced distances, and make VE′ = half of EE′ in Fig. 44, also AF = ½ AN, from Fig. 44. Then FV crosses AE′ at n the perspective of N.

For the lateral distances, make AB = NM = four feet and four inches, and draw BE′, which will be the perspective of BM, and will intersect nm, which is parallel to the ground line, and is the perspective of NM, in m, the perspective of M, the horizontal projection of the required point.

Henceforth, the two Figs. 45 and 46, differ slightly in details of construction, but not at all in principle.

In Fig. 45, the height, five feet ten inches, of the given point, is laid off on the scale of heights AN′, and N′E′ is the perspective of a perpendicular at this point. Then nn', perpendicular to the ground line, is the perspective of a vertical line at N. Hence n', the intersection of N′E′ and nn', is the perspective of the projection of M,M′ upon the lateral reference plane, AN. Then $n'm'$, the perspective of a parallel to the ground line, meets mm', the perspective of a vertical line at M, at m', the required perspective of the given point, which is thus given by three co-ordinate distances, from the three planes of reference.

In Fig. 46, the height of M is laid off on BM′. Then M′E′, the perspective of the perpendicular through it, meets the vertical, mm', at the same point, m', as was found before.

82. The method of scales can be applied to all manner of regular objects, such as buildings, etc., in simple positions, with no auxiliary drawing but a plan, on any convenient scale, laid down in any desired position oblique to the perspective plane, as indicated by the rectangular plan, in Fig. 44, and with sketches of the elevations on which the heights may be recorded.

When, however, curved surfaces, or any surfaces in quite

irregular positions, are concerned, and in finding the perspectives of shadows, generally, more or less complete projections are required, as well as the use of all the resources of perspective science.

The two following examples involve further illustrations of the method of scales.

Problem LVII.

Having an indefinite perspective vertical line, at a known distance behind the perspective plane; to find its foot; the true length of a portion of it; a given distance, in perspective, upon it; and to divide it in a given manner.

Let A*n*, Pl. V., Fig. 39, be the ground line, E*b* the vertical line, and HD the horizon.

1°. Anywhere on A*n*, make *mn* equal to the given distance of BE behind A*n*—suppose it to be one foot—and draw the perspective perpendicular, *n*C, and diagonal *m*D, through *m* and *n*. This will give T*n*, the perspective of the given perpendicular distance. Hence B, the intersection of E*b* with TB, drawn parallel to A*n*, will be the required foot of E*b*.

2°. To find the true length of E*b*, consider that, as it is vertical, any parallel horizontal lines, through E and *b*, will intercept, on the perspective plane, a space equal to the true length of E*b*. But all horizontal lines vanish in the horizon. Hence OBA, from any point, O, in the horizon, is the horizontal trace of a vertical plane through E*b*, and A*c* is its trace on the perspective plane. O*a* and O*e* are the perspectives of horizontals through *b* and E, and they intercept on A*e*, the distance *ae* equal to the original of E*b*.

3°. To find any perspective distance, as one foot, for example, above *b*. Consider that the perspectives of distances on any parallel to the perspective plane, are proportional to those distances themselves. Hence make *ap* equal one foot, and draw the perspective horizontal *p*O, which will cut off from *b* the perspective *bq* of one foot on *b*E.

4°. To divide E*b* in any desired manner, merely divide *b*E itself as required, according to the principle mentioned in the last operation.

PROBLEM LVIII.

To construct a series of vertical lines, or of parallels to the ground line, through points on a given perspective line, which divide that line into parts, perspectively equal to a given perspective distance.

Let AB, Pl. V., Fig. 38, be the ground line, and AC the given perspective line, vanishing, in this case, at C, the centre of the picture, and containing the given perspective distance P*a*. Spaces as *ab*, *bc*, etc., perspectively equal to P*a*, could be found, as in the last problem, by perspective diagonals DP and D*a*, produced to meet AB, and by diagonals through the points of division of parts equal to the original of P*a*, as found on AB. But draw the vertical PE of any convenient height, limited by a perpendicular EC. Draw *m*C bisecting PE, and draw the vertical *ae*, meeting *m*C at *n*. Then draw E*nb*, and the vertical *bf*; then *eoc*, and the vertical *cr*, etc., and *ab*, *bc*, etc., will be perspectively equal to P*a*, so that PE, *ae*, *bf*, etc., will be perspectively equidistant verticals, as required. For PE*bf* is, in space, a rectangle, hence its diagonal, E*b*, bisects its centre line *mo*, at *n*, making *ae*, in space, equidistant from PE and *bf*.

If PE be definitely given, and parallels to AB, through E,*e*, etc., be required, at intervals perspectively equal to E*e*, draw EF, and *ek*, parallel to AB, make EF equal to twice the distance E*u*, which is assumed at pleasure, and proceed as before, and as indicated in the figure.

Similar parallels can likewise be drawn in the horizontal plane.

This construction is useful in representing equidistant pilasters, and beams, or bands, in ceilings.

Should it be desired to bisect P*a*, etc., draw a line E*a*, and, from its intersection with *m*C, draw a parallel to EP, which will bisect P*a*.

SECTION V.

Inverse Perspective Constructions.

83. These constructions have for their object the determination of the real size, form, and position of an object whose perspective is given. Now, as there may be as many different perspectives of the same object, seen, too, from the same point of sight, as there may be different sections of the visual cone radiating from the eye to the object, the problem, as above stated, is absolutely indeterminate. But if we understand what the picture represents, and if the particular manner of its representation is evident, or if the ground line, or the horizontal trace of a plane parallel to the perspective plane, be known, considerable information, as to the original of the given perspective, can be obtained.

84. Thus, if any two lines of the picture are known to be parallel horizontals, their intersection will be a point of the horizon. If, in addition, they are known to be perpendiculars, their intersection will be the centre of the picture. In either case, the horizon will be a parallel to the base of the picture, through the point found. We will proceed now with a few illustrations of these principles.

Problem LIX.

Having given the perspective of a square pillar, with a face parallel to the perspective plane; to find the centre of the picture, the horizon, and the vanishing point of diagonals.

Let AB*f*, Pl. VI., Fig. 52, be the given perspective. AE and the corresponding edges being known to be perpendiculars, produce any two of them to their intersection C, which will be the centre of the picture. CD, parallel to AB, through C, will be the horizon. The pillar being known to be square, AF is a diagonal, and D, its intersection with the horizon, is the vanishing point of diagonals, and therefore CD is the distance of the eye from the perspective plane.

Problem LX.

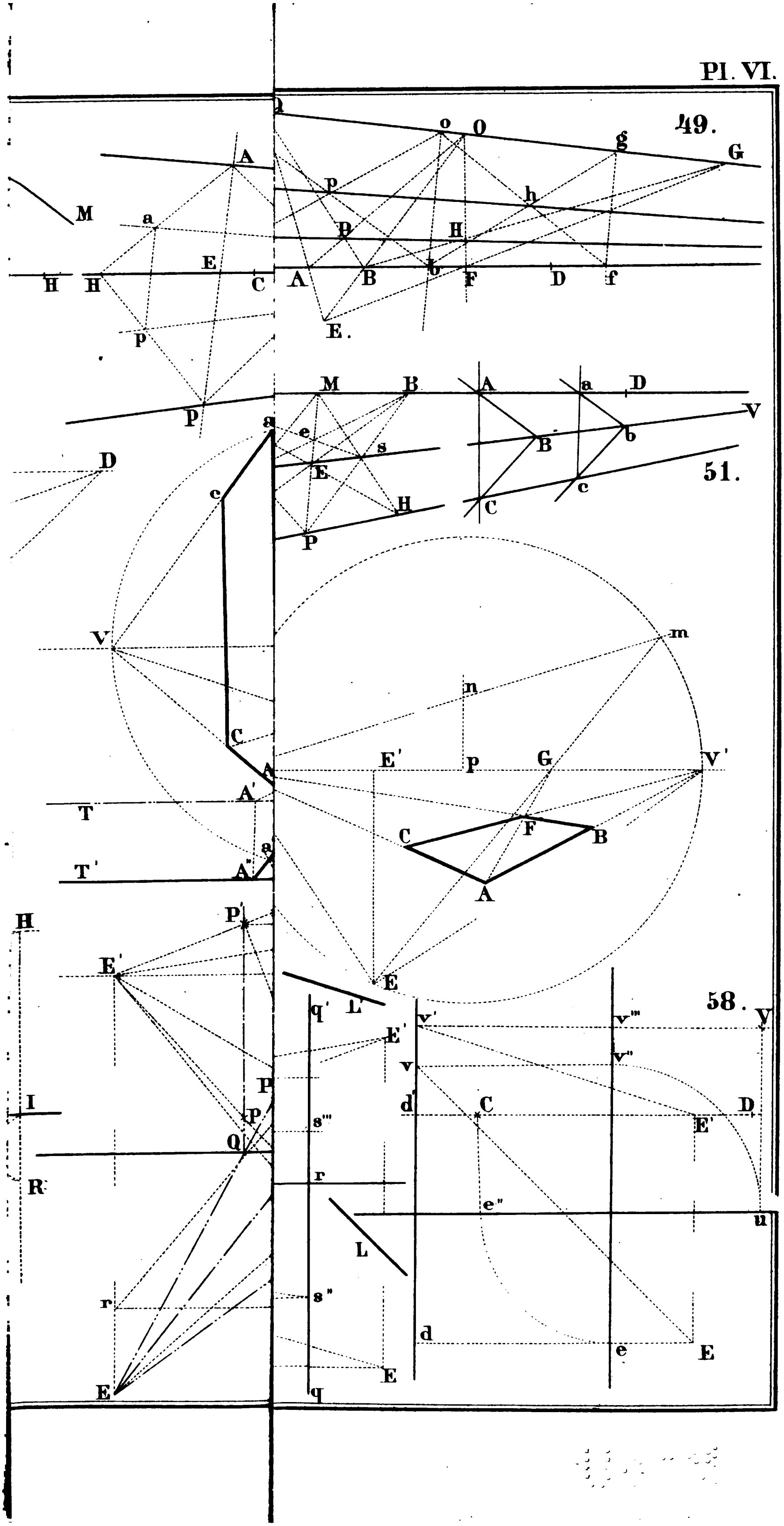
Having given the oblique perspective of the visible faces of a square pillar; to find the vanishing points of its edges, the horizon, and the point of sight.

Let ABC—a, Pl. VI., Fig. 53, be the given perspective. AB and ab, being horizontal and parallel in space, V′, their intersection is their vanishing point, and a point of the horizon. Likewise V is the vanishing point of AC and ac, and VV′ is the horizon. By drawing BV and CV′, we find F, the perspective back corner of the lower base; then AF is a diagonal of that base and has G for its vanishing point. Now, AB and AC being, in space, at right angles to each other, the visual rays parallel to them include the same angle. But these rays pierce the perspective plane at V and V′, hence the angle included by them will be inscribed in the semi-circle, having VV′ for its diameter. Besides, G is the vanishing point of the bisecting line of that angle, hence describe a circle on VV′ as a diameter, bisect its upper half at m, and mGE will be the revolved bisecting visual ray of the visual right angle sought. Therefore EE′ is the point of sight required, shown, as in previous problems, in the horizontal visual plane.

Problem LXI.

Having given, in addition to the last problem, the vertical trace of the horizontal plane containing the base of the pillar; to find the real position and size of that base.

Let A′B′, Pl. VI., Fig. 53, be the given trace, and let the true size and direction of AB, for example, be required. Produce AB to A′, in the trace A′B′; and project it into the ground line, A″B″, at A″. Then by Prob. XXV. (*Third*) revolve the parallel visual ray, EV′, into the horizon at vV′, and Ev will determine v to be the vanishing point of parallels making equal angles with A″B″ and the original of AB. Hence draw va, and vB′, through A and B, project a and B′ at a'' and B″, from which points draw $a''a'$ and B″b', parallel


49.
51.
58.

to Ev; and, from A'', the line $A''b'$ parallel to EV'. This line will intersect the two preceding ones at a' and b'; giving $a'b'$ for the real size and direction of AB. In like manner, the true size of the remaining sides of the lower base could be found.

Problem LXII.

Having the perspective of a horizontal rectangle, to find the point of sight.

Let ABFC, Pl. VI., Fig. 54, be the given perspective rectangle. Producing the pairs of opposite sides to their intersections at V and V', we have VV' for the horizon. The diagonal AF meets the horizon at G, which is its vanishing point. Now this diagonal does not bisect the angle at A, since the original figure is not a square. Hence half of VmV' will not subtend either fraction of the angle at the point of sight included between visual rays through V and V', and parallel to the originals of AB and AC, so that we only know that, if the horizontal visual plane be revolved downward about VV', the revolved position of the point of sight will be somewhere in the semi-circle on VV'. But if the ratio of the sides of the original rectangle be given, construct Vp and pn, perpendicular to each other and in the same ratio, and nVp will be equal to the angle made by the diagonal with the side of the rectangle, and $\frac{1}{2}\, mV'$ will measure this angle. Hence mGE is the revolved visual ray parallel in space to AF, and $mEV'=mVV'$ is the original size of the angle FAB; thus making EE' the required point of sight.

Problem LXIII.

To find the vanishing points of lines, in connection with the method of three planes.

The operation required by this problem is important, when finding the perspective of any body bounded wholly or largely by straight lines in parallel sets. For then all the lines of each set really will, in space, have a common vanishing point, and if they fail to do so on the drawing, it occasions the most displeasing distortion which the drawing can have. Hence *if a*

perspective cannot, on account of instrumental errors, be perfect, accuracy in other respects, must be sacrificed to secure the actual meeting of perspective parallels at their proper vanishing points.

Now in Pl. VI., Fig. 58, let *dv* be the first, and *ev'''* the second position of a perspective plane, perpendicular to the ground line, let EE' be the point of sight, and L,L' any line in space. Then E*v*—E'*v'* is the visual ray, parallel to L,L', which therefore determines the vanishing point *vv'* of L,L', and of all parallels to it. After translation and revolution, in the usual way, *vv'*, being behind the vertical plane of projection, appears at V, the vanishing point, in the picture, of all lines, parallel in space to L,L'.

Again: E*d*—E'*d'* is the visual perpendicular, which after translation and revolution gives C, the centre of the picture, and vanishing point of perpendiculars. Also a horizontal line through C is the horizon, and by making CD=E*d* (51) D will be the vanishing point of diagonals.

EXAMPLE.—*Find the perspective of any object whose lines are mostly in parallel sets, by the use of three planes, with vanishing points.*

SECTION VI.

Special Operations on Polygons and Circles.

PROBLEM LXIV.

To construct the perspective of a circle, most concisely, by means of inscribed and circumscribed squares.

The perspectives of the circular bases of cylinders have been previously found, by lines of construction from any of their points. The following method is very concise. Let CFR, Pl. VII., Fig. 63, be the given circle, and *a'e'* the ground line, the horizontal plane having been revolved 180°. Make the inscribed and circumscribed squares with sides parallel to the ground line, let D'E be the horizon, E the centre of the picture, and D' the vanishing point of diagonals. Then the lines from *a'*, *b'*, *c'*, *d'*, and *e'* to E, are the perspectives of the perpendicular sides, KA, etc., of the squares, and diameter, OH, of the circle. L'D', the perspective of the diagonal KL, then crosses the

former lines at *e*, *p*, *o*, *r*, and *k*, the perspectives of L, P, O, R, and K. Then *kg*, *rq*, *mof*, *pn*, and *ea*, all parallel to the ground line, are the perspectives of KG, RQ, etc., and cross the perspective perpendiculars, giving *h*, the perspective of H; *q*, the perspective of Q; *m* and *f*, the perspectives of M and F, etc.

THEOREM XVIII.

The middle point of the perspective of the diameter, perpendicular to the perspective plane, in a circle, whose plane is perpendicular to the perspective plane, is the centre of the perspective of the circle; and this perspective diameter, and a parallel to the ground line through its middle point, are conjugate diameters of the elliptic perspective of the circle.

In Pl. VII., Fig. 63, HC is the perpendicular diameter mentioned, and *ch* is its perspective, and a diameter of the perspective ellipse, because it has parallel tangents, *ae* and *gk*, at its

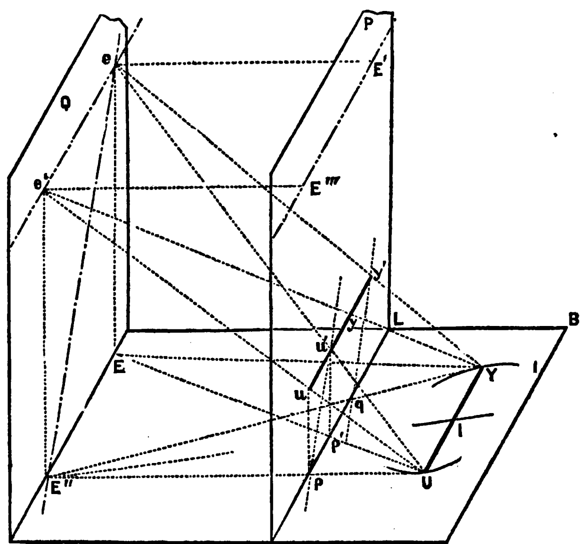


FIG. 12.

extremities; and *ch*, being a diameter, its middle point *i* is the centre of the perspective ellipse.

Now *ch*, and *uy* parallel to the ground line through *i*, will be conjugate diameters, if *uy* has tangents at its extremities, parallel to *ch*. Drawing the perspective diagonal D*it*, and its original, *t*I, the chord UY, through I, and parallel to the ground line is the original of *uy*.

Now, by Prob. XL., if the eye be in the meridian plane HC—c', the tangent at U and Y, and the diameter CH, will meet, as at E″, Fig. 12, in the horizontal trace of the visual plane parallel to the perspective plane. But, by Theor. VII., the perspectives *uy*, $u'y'$, etc., of UY are equal, and in the same line; and, by Theor. XI., the perspectives of the tangents at U and Y, and of the diameter HC, will be parallel. Now *ch* is the perspective of CH, hence the perspectives of these tangents will be parallel to *ch*, and through *u* and *y*; and will be tangents to the elliptic perspective of the circle. Hence *ch* and *uy*, each having tangents at its extremities, parallel to the other, are conjugate diameters.

Problem LXV.

Having given two conjugate diameters of a perspective ellipse, to find its axes.

Let *ab* and *xy*, Pl. VII., Fig. 64, be two conjugate diameters of the perspective of the circle AIP, and found as in the last problem. EE′ is the point of sight, Q′h', the original, and Q*h* the translated ground line; D, the vanishing point of diagonals, and *o* the centre of the ellipse, whose original, O, may be found as in the last problem, giving *gf* for the original of *xy*.

Otherwise: by Theorem VIII. the perspective of *gf* will be equal and parallel to *xy*, wherever, in the vertical plane on Q*h*, the eye be taken. Hence transfer EE′ for a moment to F,E″ and the tangents to AIP, from F, will determine *gf*, and thence, O.

Proceeding now to find the originals of the axes, the following conditions must be met:

1°. Their perspectives must pass through the centre *o* of the ellipse.

2°. These perspectives, both on the pláne Q′h', and on Q*h*, taken now as an auxiliary perspective plane, must be at right angles to each other.

3°. These perspectives, to be not only rectangular *diameters*, but rectangular *conjugate* diameters, and therefore *axes*, must have tangents at the extremities of each, parallel to the other.

To fulfil the *first* condition, the originals of the axes must be chords of AIP, passing through O. To fulfil the *second*, their perspectives must be parallel to the traces of visual planes through them, upon the vertical visual plane Q*h*. And these traces being at right angles to each other, must be inscribed in a semicircle, through E′, and with its diameter on Q*h*. To fulfil the *third*, both of the two required chords, as HG, supposing it for a moment to be one, and the tangents at the extremities of the other, as IP, which are to be parallel tangents in perspective, must, by Theorem XI., concur at points on the horizontal trace, Q*h*, of the vertical visual plane; and these points will be the extremities of the diameter of the semicircle, in which the traces, above mentioned, must be inscribed.

Next, to find two such points, so that the desired rectangular traces from them to E′ can be found.

Conceive any two points of Q*h*, as M and N, to be the centres of two spheres. The arcs, as GH and IP, of the great circles cut from these spheres by the horizontal plane, intersect at K, a point of the circle of intersection of these spheres. Also, the vertical small circles, HG, and IP, of these spheres, intersect in a vertical chord at O, hence KO is the horizontal projection of an arc of the circle of intersection of the spheres. But by the properties of surfaces of revolution, this circle must be perpendicular to the common axis, MN, of the spheres; that is K is on AB, and O being the middle of *fg*, the vertical chord, at O, of AIP, considered as a sphere, is equal to *fg*. Hence the arc *fg* is equal to the one of which KO is the horizontal projection, and it therefore contains K. Hence, as M and N have been treated simply as *any two* points, of Q*h*, we conclude that *any* sphere with its centre on Q*h*, and containing a chord of AIP, through O, would contain K. Hence we may conceive the required semicircle through E′, before metioned, to be the intersection of the plane Q*h* with one of these spheres. Then this semicircle would, after revolving back the plane Q*h*, coincide with the horizontal great semicircle of the same sphere, and we have only to find a semicircle through E′ and K, with its centre on MN. Its centre is at once found at L by drawing *k*L, perpendicular to

the middle point of E′K. Then the semicircle with radius LE, determines the points M and N. By Theorem XI., ME′ is the perspective, on the auxiliary perspective plane Q*h*, of the secant and tangents which concur at M, and NE′ is that of the secant and tangents which concur at N. Now ME′ and NE′, being inscribed in a semicircle, include a right angle, and as the two perspective planes are parallel, the required perspectives of the lines meeting at M and N are parallel to ME′ and NE′. These perspectives, for the latter set, begin at *m*, *e*, and *h*, translated positions of *m*′, *e*′, and *h*′ (regarding Q*h* again as the translated position of Q′*h*′), hence they are *hs*, *e*U, and *mr* parallel to NE′. U*e* is the indefinite transverse axis of the perspective ellipse, and the parallel tangents, *hs* and *mr*, limit the conjugate axis *rs*, perpendicular to U*e*, through *o*, at *r* and *s*. The line *rs*, and the parallel tangents at U and T could be found as just explained, but the indefinite axes may be limited by perspective perpendiculars, as at U, by *u*E, the perspective of the perpendicular from P; or by perspective diagonals, as at *s*, by *t*D the perspective of the diagonal from G.

Remark.—Reviewing this problem, we see that the actual construction is brief and simple. The greater extent of a full explanation of it, may account for its being seldom met with.

THEOREM XIX.

The perspective of a circle may be any one of the conic sections.

For the eye is the vertex of the visual cone, whose base is the given circle, and the perspective plane may cut this cone in any manner whatever. If it cut all the elements, the section will be an ellipse, except when the plane is parallel to the base, or, in the case of an oblique cone, cuts it in a subcontrary circular section. (See Des. Geom. .) If the perspective plane be parallel to a tangent plane to the visual cone, it will cut it in a parabolic perspective of the circle; and if it cut both nappes, the perspective will be a hyperbola. The two latter forms correspond to the case in which the perspective plane cuts that of the circle, which is quite unusual in practice.

EXAMPLE.—*Construct the perspective of a circle, which is tangent to the visual plane, parallel to the perspective plane.*

85. Pl. X., Fig. 74, shows a familiar construction of the parabola, upon a given chord and segment of the axis, by means of equidistant parallels to the latter, meeting radials from the vertex *c* to equidistant points on the outer parallel K*d*, which is equal to *c*H. Fig. 77 shows at C, A′, B′, etc., a familiar construction of a circle by points, VE and ED being equal, and equally divided by the radials from C and H, meeting VE and DE produced, if necessary, The principles of perspective now permit the solution of the following problem, and then it will be shown how the above construction of the parabola is derived from that of the circle.

Problem LXVI.

To construct the parabolic perspective of a circle.

We shall make an exact construction of the horizontal projection of the perspective of the circle, which will, however, be a proper representative of the perspective itself, since that projection of a conic section is a conic section still, of the same kind. Then let CEHF, Pl. X., Fig. 77, be the given circle, constructed by points, and let it be the base of a vertical right cone, of which V is, therefore, the vertex, and VC and VH two opposite elements, in horizontal projection. Let GL be the horizontal trace of the perspective plane, which, to contain a parabolic perspective of the circle, must be parallel to one of the elements just named, as VH. Then to find the intersection, *c*, of this plane with CV, or the visual ray from C, consider the elevation of a cone in Fig. 75, similar letters denoting similar points in the two figures. L*c*, being parallel to VH, we have

$$\text{CH} : \text{CL} :: \text{CV} : \text{C}c, \text{ and}$$
$$\text{CV} : \text{C}c :: \text{C}v : \text{C}q,$$
$$\text{therefor CH} : \text{CL} :: \text{C}v : \text{C}q.$$

Hence make C*c*, Fig. 77, equal to C*x*, a fourth proportional to CH, CL, and CV, and *c* will be the intersection required, that is, it will be the perspective of C. All lines meeting at V, are projections of visual rays; hence, CD, being parallel to the perspective plane, *cd*, limited at *d* by the ray DV, is the perspective of CD. But G, the intersection of DG with the perspective plane, is its own perspective, therefore G*d* is the

perspective of GD, and the intersections of it, as a, with visual rays, as AV, from the points of division on GD, are the perspectives of those points. But the lines of construction from A, etc., meet at C, whose perspective is c, hence ac, bc, etc., are the perspectives of AC, BC, etc. Now the visual ray, VH, being parallel to the perspective plane, the perspective of H is at an infinite distance from GL, and hence the perspectives of the lines, A′H, etc., which concur at that point, are parallel. But Ve is the perspective of VE, and both are similarly divided, since VE is parallel to the ground line; hence n 1′, o 2′, etc., parallel to VH, are the perspectives of AH, etc.; and A″, B″, etc., their intersections with ac, bc, etc., are the perspectives of A′, B′, etc., and are points of the parabolic perspective, cA″M″, of the given circle. M is found by making Ek=EM, and drawing Hk and CM. Then the visual ray MV gives m, the perspective of M, and mc, that of MC; and, making e 4′=2′3′, we have 4′M″, parallel to VH, for the perspective of H,M′, and M″, its intersection with Gd, for the perspective of M′.

The way is now open to the proposed theorem (85).

THEOREM XX.

The construction of the parabola by points (Fig. 74) *is the perspective of the given construction of the circle, by points* (Fig. 77).

To prove this theorem, it is only necessary to show, that the system of lines from c, divides any one of the parallels to VH, into equal parts. This is easily shown by reference to Pl. X., Fig. 76. There, NDG represents a vertical plane of which DG (Fig. 77), is the horizontal trace. Nd″PG represents the perspective plane, and NDCc, KACc, etc., the visual planes, whose horizontal traces, DC, etc., represent DC, etc., in Fig. 77. Then Nc, Kc, Rc, etc., represent the indefinite perspectives of DC, AC, etc. Now, by reason of the common intersection Cc, of the visual planes, their traces DN, AK, etc., on the supposed vertical plane DG, are parallel. Hence, DA, AB, etc., being equal, NK, KR, etc., are equal; therefore, as d″M″, which replaces d′M″, Fig. 77, is parallel to NG, the

divisions $d''a''$, $a''b''$, etc., are equal, and hence, finally, as $e''e'$, $b''b'$, etc., are obviously parallel, $d'a'$, $a'b'$, etc., are equal. But these last spaces represent $d'a'$, etc., in Fig. 77. Hence the latter distances are equal, which justifies the construction of Fig. 74.

EXAMPLE 1.—*Find, upon the vertical perspective plane, the parabolic perspective of a circle one element of whose visual cone should be—vertical or otherwise—in the visual plane parallel to the perspective plane.*

Ex. 2.—*Find the original of the axis, focus directrix and any ordinate of this parabolic perspective.*

Ex. 3.—*Repeat Ex.* 1 *and* 2; *for the hyperbolic perspective of a circle whose visual cone is cut by the visual plane, parallel to the perspective plane.*

PROBLEM LXVII.

To construct the perspective of a circle, without drawing the circle, and by means of the numerical relations existing among the sides and diagonals of the circumscribed square.

In common to all the following methods, first construct the perspective of the circumscribed square, one of whose sides is supposed to be parallel to the ground line. To do this, it is only necessary to make $a'e'$, Pl. VII., Fig. 63, equal to the side of the square, and $e'L'$ equal to $e'L$, the distance of the original square from the ground line, and then to complete the perspective square $aegk$ by perpendiculars, and a diagonal, as seen in the figure. Then:

First. Make $d'e'=a'b'=\frac{1}{7}$ of $a'e'$, to locate the perpendiculars which give p and r, and thence n and q. For,

$$d'e'=\mathrm{FT}=\mathrm{OF}-\mathrm{OT}$$

But OT=cos 45°, for the radius OF=R

$$=\mathrm{R}\sqrt{\tfrac{1}{2}}=\mathrm{R}\sqrt{\frac{24\frac{1}{2}}{49}}=\tfrac{5}{7}\mathrm{R} \text{ very nearly.}$$

$$\text{Or } d'e'=\mathrm{FT}=\tfrac{2}{7}\mathrm{R}=\tfrac{1}{7} \text{ of } a'e'.$$

This method is, therefore, a close approximation.

Second. Draw em, after drawing fm parallel to the ground line through o, the perspective of the centre O. Then make

e′s′ equal to $\frac{1}{5}$ of *a′e′*, and draw *s′*E, whose intersection, *s*, with *em*, will be a point of the perspective of the circle. Three other points can be similarly found.

Here we have $LS \times LM = (LC)^2 = R^2$

But $(LM)^2 = (LA)^2 + (AM)^2 = 5\ R^2$

Hence $LS \times LM = \frac{1}{5}\ (LM)^2$

Or $LS = \frac{1}{5}\ LM$, and $Lx = e's' = \frac{1}{5}'\ LA = \frac{1}{5}\ a'e'$.

Third. Pl. X., Fig. 77, shows at A′, B′, etc., a familiar construction of the circle by points, the radius VE and the equal tangent ED, perpendicular to it being equally divided by the groups of lines meeting at C and H, meeting in pairs at A, B, etc. The perspectives of these points, found in any convenient manner, will be points on the perspective of the circle, through C, A, B, E, H, etc.

Fourth. Make $L1 = \frac{1}{6}$ of LA; and the intersection of 1F and LH, both of whose perspectives are readily found, will be a point, V, of the circle. For, let V3 be parallel to LA, then L3 : LG :: LV : LH. But by the second method

$$LV = \tfrac{1}{5}\ LH, \text{ hence } L3 = \tfrac{1}{5}\ LG \text{ or } \tfrac{2}{5}\ R,$$

$$\text{or } F3 = \tfrac{3}{5}\ R \qquad \text{and} \quad V3 = \tfrac{1}{5}\ HG = \tfrac{1}{5}\ R.$$

Therefore $V3 = \frac{1}{3}$ of F3 which makes

$$L1 = \tfrac{1}{3} \text{ of } LF = \tfrac{1}{6}\ LA, \text{ as in the construction.}$$

Fifth. Make $A2 = \frac{1}{8}$ AL, and 2K will meet AH′ in a point X, of the circle; whose perspective, as before, will be the intersection of the perspectives of 2K and AH. For, the similar triangles, A2X and KXH, give

$$AX : XH :: A2 : KH$$

$$\text{But } AX = \tfrac{1}{5}\ AH = \tfrac{1}{4}\ XH, \text{ hence } A2 = \tfrac{1}{4}\ KH = \tfrac{1}{8}\ AL.$$

It will easily appear, that by taking LM in place of LH or AF in place of AH, that each of the last two methods will give *two* points in each quadrant, and that these two methods give the *same* two points.

Other like methods could be given, but the above, being the best, are enough for illustration. The practical convenience of all of them depends on having the square with a side parallel to the ground line.

PROBLEM LXVIII.

To divide a circle into parts perspectively equal; and with, or without, recourse to its plan.

The parts of the original circle will be really equal. Then, in Pl. VII., Fig. 63, for example, divide the circle HFM into equal parts by radii which shall be produced to meet the sides of the circumscribing square. Lines joining the perspectives of the latter points with the perspective centre, o, will divide the perspective circle as required.

This problem, being useful in the construction of perspectives of toothed wheels, fluted columns, or circular colonnades, the following construction, independent of the plan, is given. Describe a semicircle on mf as a diameter, and divide it into equal parts. Draw ordinates to mf through these points of division, and from their feet draw lines to E. The latter lines will divide the entire perspective circle as required, since the semicircle is the original, in a plane having mf for its ground line, of a circle whose diameter is mf, and the ordinates are horizontal projections of perpendiculars.

Remark.—The method of chords Prob. XXV. (line ub) may be applied to this problem, when the circumscribing square has any position. Also in general, *any* system of parallels being drawn, through equal divisions of the original circles, their perspectives will have a common vanishing point, and will divide the perspective circle into the perspectives of the equal parts.

PROBLEM LXIX.

To find the perspective of a regular hexagon, without constructing its hexagonal projection.

Let Ak, Pl. VII., Fig. 65, be the ground line, DV′ the horizon, EE′ the point of sight, Table , and D the vanishing point of diagonals; E′D being equal to E′E. Let AB be equal to one side of the hexagon; and At, equal to its distance behind the perspective plane. Then the perspective perpendicular, AE′, and diagonal, tD, intersect at a, the perspective of a

vertex of the hexagon. Supposing one side to be parallel to the ground line, then ab, limited by BE′, will be its perspective. The next four sides, from a and b, make angles of 60° with the ground line; hence draw EV, so as to make the angle at E=30°, make E′V′—E′V, and V and V will be the vanishing points of these four sides and of their parallel diagonals. Thus bV and the perspective perpendicular AE′ determine e; then V′e and aV intersect, giving f, through which fc, the diagonal of the figure, parallel to the ground line, meets bV at c, another angle of the hexagon. Lastly, cV intersects ed, parallel to the ground line, at d, completing the figure.

It may be noted that fc, being parallel to the ground line, is bisected at o, by bV just as its original is, by that of bV. Also on evidently equals nc. Hence, again, if Bk be made equal to half of AB, the line kE′ will contain the point c, at its intersection with bV′ or fc.

Problem LXX.

To find the perspective of a regular octagon, without making use of its plan.

Let the ground line, and given vanishing points be as shown in Pl. VII., Fig. 66, and let AB be equal to one side of the octagon, and At, its distance behind the perspective plane. Then, the perspective perpendicular, AE, and diagonal, tD, will determine a, the perspective of an angle to the octagon; and ab, parallel to the ground line, and limited by BE, will be the perspective of that side of the figure, which is supposed to be parallel to the ground line. Now, by inspecting an octagon and the circumscribing square containing its sides, it will be seen that the distance from an angle of the octagon to the adjacent corner of the square will be a distance, Bk, the hypothenuse of a right triangle each of whose other sides equals half the side AB of the octagon. Then, as shown, make Bk' and Ak'' each equal to Bk, and k'E and k''E will be, in perspective, sides of the circumscribing square. Four sides of the octagon, two of which are adjacent to ab, are diagonals, in parallel pairs, to which diagonals of the octagon are also parallel. Hence draw bD, and aD will limit the side ah at h, on k''E ; bD limits

hg at *g*, on *k''*D; *hc*, parallel to the ground line, determines *c*, on *k'*E; then *c*D meets AE at *f*, giving *gf*; *gd* determines *d*, on *k'*E; and, finally, *e* is found as the intersection of *fe*, parallel to AB, with BE, or of *d*D with *fe*, or BE.

EXAMPLE 1.—*Let the perspective of a regular pentagon now be constructed, as an exercise, without making its plan, having given the perspective of one of its sides, and the point of sight.*

CHAPTER V.

Elementary Perspectives of Shadows.

86. In this chapter, it is proposed to state, and illustrate, in an elementary manner, the methods of finding the perspectives of shadows.

There are two general methods of finding perspective shadows. *First;* by *separate* successive applications of the principles of shadows, and of perspective. *Second;* by the *combined* application of these principles.

87. According to the *first* method, the projections of shadows desired, are found in the usual ways, then the perspectives of these shadows are found, just as the perspectives of the objects casting them are.

88. According to the *second* method, knowledge of the position, form, and properties of a given body, and of the principles of shadows, and of perspective, permits the direct construction of perspectives of shadows without reference to their projections.

89. Any vertical surface will be intersected by a vertical plane of rays in a vertical line, which will meet any ray in the latter plane, in the shadow of the point through which the ray was drawn. Also the shadow of a point upon the horizontal plane is the intersection of a ray, through that point, with its own horizontal projection; and the *perspective* of that shadow will be the intersection of the *perspective* of the ray with the *perspective* of its horizontal projection; and the like is true for any other plane. In both cases, moreover, the horizontal trace of the vertical plane through the given ray is the horizontal projection of that ray and of all rays in that plane. Hence, great use is made, in perspective, of the perspectives of rays, and of their projections on given planes, simply located; and it therefore becomes necessary to find the vanishing points of these lines.

The foregoing principles, with other subordinate ones, are illustrated in the following problems.

Problem LXXI.

Having given the point of sight, and the direction of the light; to find the vanishing point of rays, and of their horizontal projections. Also, having these points given, to find the direction of the light.

First. Let EE′, Pl. VI., Fig. 55, be the point of sight, A and A′, the projections of a ray of light, and QI the ground line. By the rule for vanishing points, EI—E′R; the visual ray parallel to A,A′, and piercing the perspective plane at R, gives R for the vanishing point of rays; so that every line drawn from R is the perspective of some ray parallel, in space, to A,A″.

Again, EI—E′H is a visual ray, parallel to the horizontal projections of all rays parallel to A,A′. But EI—E′H pierces the perspective plane at H, which is therefore the vanishing point of horizontal projections of rays.

The two visual rays, EI—E′R and EI—E′H, are obviously in the same *vertical* visual plane. Hence the vanishing points, R and H, are in the vertical trace, RH, of that plane, which is perpendicular to the ground line, and is by (42—3°) the vanishing line of all vertical planes of rays parallel to A,A′. Further, as horizontal projections of rays are horizontal lines, their vanishing point, as H, is necessarily in the horizon (33—2°).

Second. Conversely, therefore, having given the point of sight, EE′, and the vanishing points, R, and H, *to find the real direction of the light.*

Join I, the intersection of the ground line with the vertical trace of the vertical visual plane, EIH, with E. This will be the direction of the horizontal projection of the light. Join R with E′, and that will give the vertical projection E′R of a ray of light.

We can thus either assume the direction of the light, and find the vanishing points of rays and of their horizontal projections; or the reverse.

Problem LXXII.

To find the perspective of the given shadow of a given point, on the horizontal plane.

Let P,P′, at a distance Pq, Pl. VI., Fig. 55, behind the perspective plane, be the given point. The ray, PS—P′S′, through P,P′, pierces the horizontal plane at S, giving S as the shadow of P,P′. Then, by the method of visual rays (Prob. V), or of vertical visual planes and perpendiculars, which are identical in construction (53) s is the perspective of the point of shadow, S,S′. Thus SE, and S′E′, are the projections of the visual ray from the shadow S,S′; and this ray pierces the perspective plane at s, which is therefore the perspective of S,S′, as required.

Problem LXXIII.

To find the perspective shadow of a given point, on the horizontal plane, without having its real shadow.

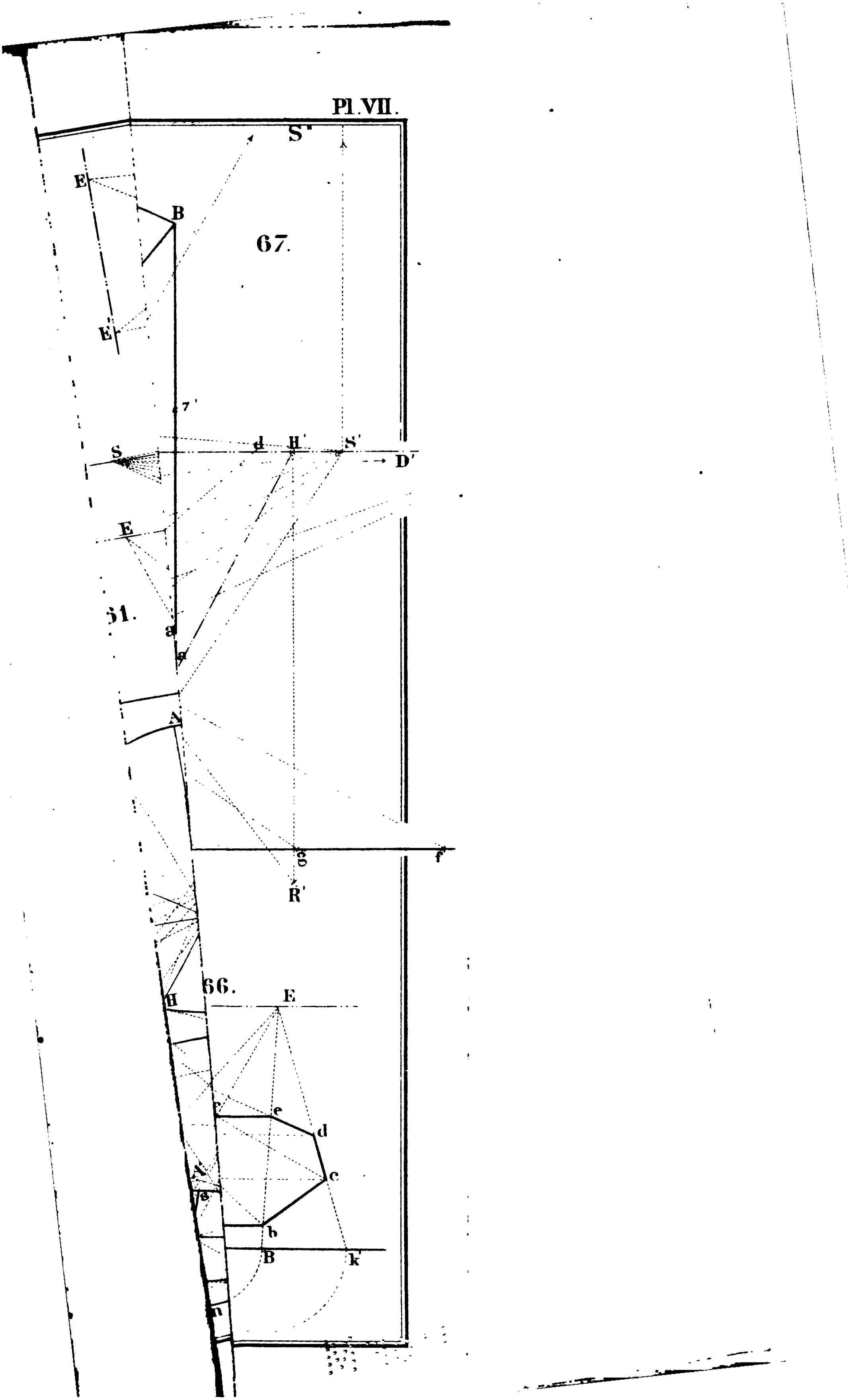
Continuing to refer to Pl. VI., Fig. 55, p' is, by either of the methods of the last problem, the perspective of the given point, P,P′, in space; and p is the perspective of its horizontal projection, P,q. Then p'R is the perspective of a ray through PP′, and pH is the perspective of the horizontal projection of the same ray. Hence s, the intersection of p'R and pH is (89) the perspective of the shadow of PP′ on the horizontal plane.

Problem LXXIV.

To find the perspective of the shadow of a point, on a plane perpendicular to the ground line; by either general method.

Let PP′, Pl. VI., Fig. 56, be the given point; QI, the ground line; EE′, the point of sight; and R, the vanishing point of rays. Let the plane SYS′ be the one on which the shadow of PP′ is to be found.

Now by the *first method* (87) S and S′ are the projections of


S'
E
B
67.
E'
7'
S
d
H'
S'
D'
E
61.
a
a
A
g
f
R'
66.
H
E
c
e
d
c
A
a
b
B
k'
n

the intersection of the ray of light, PS—P′S′, with the plane SYS′, and hence, are the projections of the shadow required. Then the visual ray SE—S′E′ pierces the perspective plane at *s*, making *s* the perspective of the shadow S,S′, cast by P,P′ on the plane SYS′.

By the *second method*, find, first, *p*′, the perspective of PP′. Then note that P‴ and P″ are the projections of the projection of PP′ on the given plane SYS′, and *p*″ is the perspective of this projection; found like *p*′, by the method of visual rays, or in any other convenient way. Note that *p*′*p*″, being the perspective of the projecting line PP‴—P′P″, which is parallel to the ground line, is, itself, parallel to the ground line; so that *p*″ can be found as the intersection of *p*′*p*″ with U*p*″, the vertical trace of the visual plane P‴—EU*p*″. Now, the visual ray, parallel to the projections of rays on SYS′, will evidently meet the perspective plane in the line E′*r*. But this ray, and the visual ray, E′R, are in a plane perpendicular to SYS′, that is, parallel to the ground line, hence R*r* is its trace on the perspective plane; and *r* is the vanishing point of projections of rays on the plane SYS′. Then the perspective ray of light, *p*′R, meets *p*″*r*, the perspective of its projection on SYS′, at *s*, the perspective of the shadow of PP′, that is of *p*′, on the plane SYS′, as before found.

Problem LXXV.

To find the perspectives of the shadows of any points, upon any oblique plane whose traces meet in the ground line.

Let M,M′ and N,N′, Pl. XV., Fig. 103, be two given points, and let PQP′ be a given plane; and let it be required to find the perspectives of the shadows of these points upon this plane.

Observing the kind of lines used in the construction, we see that E*d*M, and *d*M″, are the horizontal and vertical traces of a vertical visual plane through M,M′; and M′E′ is either the *vertical projection of the visual ray* from M,M′, or the *perspective of the perpendicular* through the same point (53). In either view M″, is the perspective of M,M′, and *m*″ is the perspective of M, the horizontal projection of M,M′.

By similarly situated lines, inked however as projections of a visual ray, and projecting lines of a point, N″ is found to be the perspective of N,N′, and n'', that of its horizontal projection. This completes the perspectives of the points.

The vertical trace of the given plane is its own perspective. The perspective of its horizontal trace contains Q, where it pierces the perspective plane (Theor. III.) and also contains its vanishing point. But this trace being a horizontal line, its vanishing point is in the horizon (42—4°) and is at p', where the visual ray Ep—E′p' parallel to the trace PQ, pierces the perspective plane. Then Qp' is the perspective of the horizontal trace, PQP″ of the given plane.

Now, the vanishing line, p'I, of the given plane is drawn through p', parallel to its vertical trace P′Q (Theor. IV.) and this vanishing line necessarily contains the vanishing points of all lines in the given plane. Therefore, to find the required shadows, we proceed as follows.

Assume R, the vanishing point of rays, and H, that of their horizontal projections; or construct them if the direction of the light is given; then Hm'' is the perspective of the horizontal trace of a vertical plane of rays through M,M′, and, as this plane is vertical, RHI is its vanishing line. Now the intersection of this plane with the given plane, contains the shadow of the given point M,M′; but this intersection being common to the two planes, its vanishing point is in the vanishing line of each plane, and hence at I. Also a, the intersection of the perspective horizontal traces of these planes is another point of the perspective of their intersection, which is therefore the line aI. Then the perspective, M″R, of a ray of light through M,M′ meets aI at S, the perspective of the shadow of M,M′.

Note also that G′, being the intersection of the vertical traces of the two planes, is its own perspective; that is, the line Ia passes through G′.

T, the perspective of the shadow of N,N′ upon PQP′ is similarly found, and if joined with S, would give the shadow of a line, NM—N′M′, upon PQP′.

Again; s is evidently the perspective of the intersection of the vertical line M—M′m', whose perspective is M″m'', with the given plane. Hence sS is the perspective of the shadow of this vertical line upon the plane PQP′.

To verify the construction, note that MG, being the actual

horizontal trace of the vertical planes of rays through M,M′, it will be parallel to the horizontal projection E*h* of a visual ray *of light;* also that *a* is the perspective of A, the original intersection of the real horizontal traces of the two planes.

Examples.—1. *Let the angle P′QL be obtuse, and let R be above H.*

2. *The same conditions remaining, let M,M′, be taken in the Hor. Plane.*

3. *Let PQL be a right angle, and let M,M′ be in the first angle.*

4. *Let P′QL be a right angle, and let M,M′ be to the right both of PQP′ and of the line RH.*

Problem LXXVI.

To find the perspective of the shadow of a point, upon any oblique plane, which is parallel to the ground line.

Let M,M′ be the given point, Pl. XV., Fig. 104, and let the plane be determined by the two parallels to the ground line, *hc—h′c′* and *eb—e′b′*. GL is the ground line, and E′H, the horizon. Any secant, *cb—c′b′*, which intersects both of these lines, will intersect the planes of projection in points of the traces of the oblique plane, which are therefore the lines PQ and P′Q′, parallel to the ground line. This plane being parallel to the ground line, its vanishing line will be so also, and one point of it will be where a visual ray, as E*p*—E′*p*′, parallel to any line, as *cb—c′b′*, in the plane, meets the vertical, that is, the perspective plane, that is at *p*′. Hence *p*′I is the vanishing line of the given plane.

The perspective of M,M′, found as in the last problem is M″, and *m*″ is the perspective of its horizontal projection. R and H are, as in the last problem, the assumed or constructed vanishing points of rays, and of horizontal projections of rays, respectively, and RHI, perpendicular to the ground line, is the vanishing line of a vertical plane of rays through M,M′, and HG, drawn through *m*″, is the perspective of the horizontal trace of this plane. Next, *g*′ being the perspective of Q,*q*′, a point of the horizontal trace, PQ, of the given plane, *ag*′, parallel to the ground line, is the perspective of PQ; and *a*, its

intersection with HG, is a point in the perspective of the intersection of the vertical plane of rays with the given plane. This line, being thus common to both planes, its vanishing point is I, the intersection of the vanishing lines of the two planes; and aI is its perspective. Then S, the intersection of the perspective ray M″R, with aI, is the perspective of the shadow of M″ upon the given plane PQ—P′Q′.

As in the last problem Ia passes through P′, which evidently corresponds with G′ in the preceding figure. Also, as before, sS is the perspective of the shadow of M″s upon the given oblique plane.

EXAMPLES.—1. *Let this problem be solved when the given plane, still parallel, in all these examples, to the ground line, crosses the second angle.*

2. *Also, when M,M′ is to the right of E,E′.*

3. *Also, when M,M′ is to the right of both E,E′ and H.*

4. *Also, when M,M′ is in the vertical plane, as at m′,M′.*

5. *When R is above H.*

PROBLEM LXXVII.

To find the perspective of the shadow, cast by a point, on the horizontal plane, by the method of three planes.

In this case, care must be taken to remember that the projections of the light must be taken with reference to the observer as facing the perspective plane, so that on the perspective figure the shadow shall, as usual, indicate the light as coming obliquely over the left shoulder.

Thus, let P,P′ be a given point in space, Pl. VI., Fig. 57; QRQ′, the perspective plane, and E,E′, the point of sight. Then, as the observer, standing at E, looks at the perspective plane in a direction parallel to the ground line, PS and P′S′ are the proper projections of the light, and S,S′ is the shadow of P,P′ on the horizontal plane. Then by the usual operations in case of three planes, translating QRQ′ to qrq', and revolving it; S″ is the perspective of SS′.

THEOREM XXI.

If the perspective of a ray of light is tangent to the perspective of a curve, it will also be tangent to the perspective of the shadow of that curve upon any surface.

The perspective ray, R, being tangent to the perspective of a curve, C, shows that it is the trace of a plane, tangent to the visual cone, of which the original of the curve is the base.

But such a plane is a *visual* plane, and it will be tangent to the cylinder of rays of light whose given base is the same given (original) curve. Hence its trace, T, on any surface, will be tangent to the intersection of this cylinder of rays with that surface; that is, to the shadow, S, of the given curve in space. But the tangent plane, being also a visual plane, its trace on the perspective plane, which is the given perspective ray, R, is the perspective of the tangent trace, T; and by (Theor. VI.) this perspective of T, will be tangent to the perspective of the shadow, S; which was to be proved.

CHAPTER VI.

General and Practical Problems of Forms and their Shadows.

90. In this chapter of larger and more practical applications, of the methods already explained, those methods will, in general, be applied indiscriminately, several different ones being sometimes employed in the same problem, so that particular methods shall not become unnecessarily associated with particular problems. Only, in the construction of certain special points, the methods most favorable to accuracy will be chosen.

PROBLEM LXXVIII.

To find the perspective of an interior, and of one of its shadows.

Let ABCG, Pl. VII., Fig. 67, be the section of an arched room, made by the perspective plane, whose ground line is AG. This section is therefore its own perspective. EE′ is the point of sight, taking the plane of the horizon for a plane of projection; and D and D′ are the vanishing points of diagonals. The other vanishing points will be explained as they shall come to be used. This problem will serve chiefly to illustrate the method by scales. See Prob. LVI.

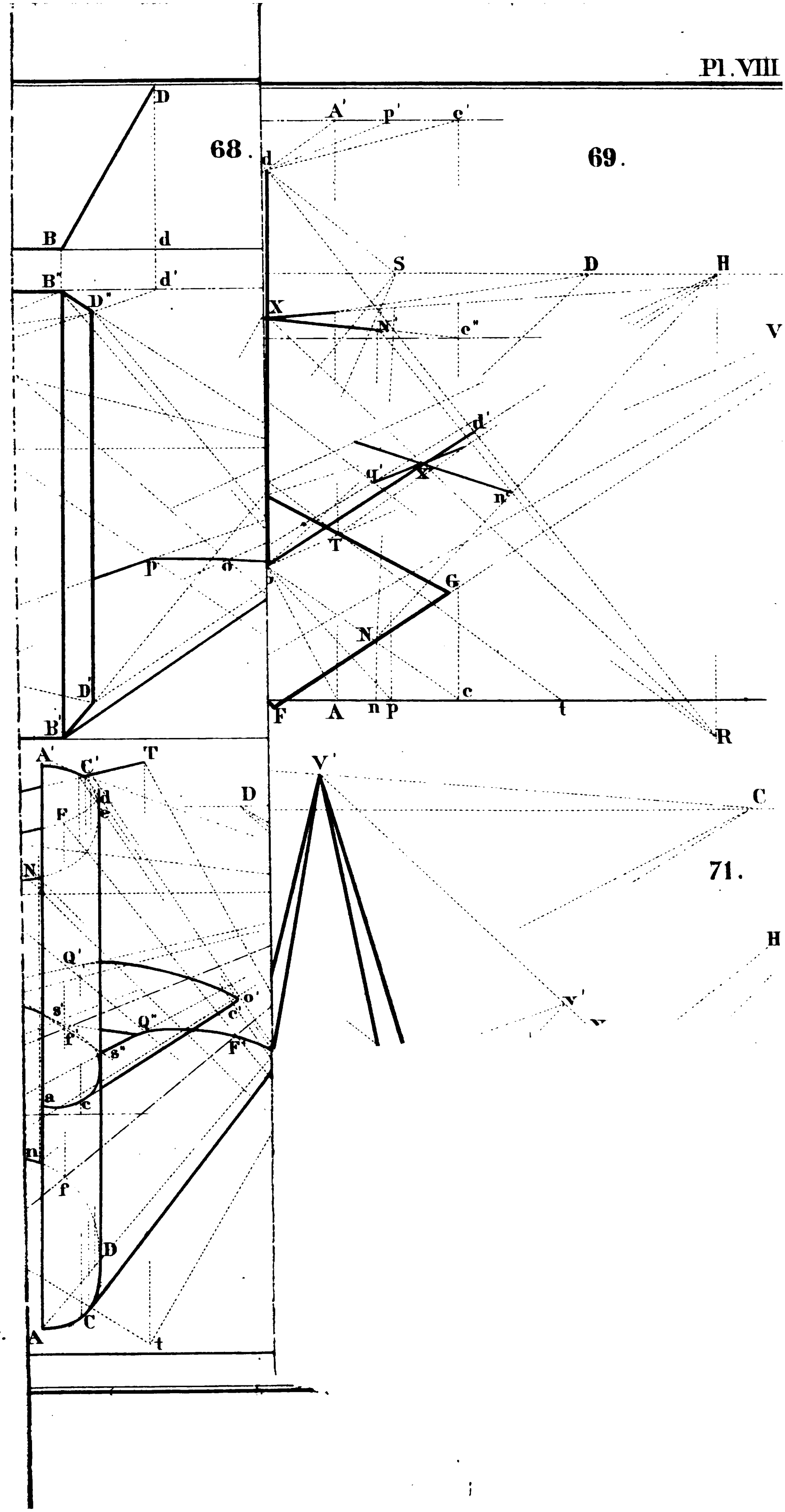
1°.—*To find the perspectives of the internal edges of the room.* Four of these edges, from the points A, B, C, and G, are perpendicular to the perspective plane, and therefore vanish at E′. Let the room be 12 ft. in depth back of the perspective plane, which may be supposed to coincide with one end of the room. Then, making A—12=12 ft., on the scale of AG=8 ft., as the width of the room, 12—D′ will be the diagonal which will limit AE′ at K, a back corner of the floor. Supposing the rear wall of the room to be parallel to the perspective plane, draw KH, which will limit GE′ at H; KJ, which

will limit BE′ at J; and HI, which will limit CE′ at I. Let the centre, c', of the arc, BC, of the ceiling, be the vertex of an angle Bc'C of 30°; then to find the perspective IJ of the equal and parallel back arc, draw $c'c$, parallel to AG, then ch—E′, the perspective of a line parallel to GH, then hh', parallel to AG, which will meet gh' at the centre of IJ. And gh' is found by dropping $c'4$, perpendicular to AG, and drawing $4q$E′, the perspective horizontal trace of a vertical plane, through the centres c' and h', upon the floor, then qh' perpendicular to KH, is the trace of the same plane on the rear wall. The arc IJ, with centre h' then completes the outline of the interior.

2°.—*To find the perspective of the doorway and pedestal.* Let the hither edge of the doorway be 4 ft., from G in the left-hand wall, then, G12 being 4 ft., the perspective diagonal 12—D′ cuts off the perspective, GL, of four feet on GH. Next let the doorway be 3 ft. wide, so that its further edge shall be 7 ft. from G; then 1—D will be a perspective diagonal giving GM, the perspective of 7 ft. Also, let the door be 8 ft. high; then, on GC, make G8=8 ft., and 8—E′ will limit vertical lines, at L and M, so as to give the upper corners of the opening. If the wall be 1 ft. thick, 9—E′ will be the perspective of its outer floor line, and MP, and OQ, will be parallel to AG, and PQ will be vertical. A fragment of the outer top edge is seen at Q, vanishing at E′, it being, in space, parallel to NO.

Passing to the pedestal, we suppose it to be 3 ft. high, and 1 ft. square; 1 ft. back of AG, and 1 ft. to the left of the right hand wall. Hence 2—E′ and 1—E′ are the perspective traces of its vertical side faces. The diagonals, 7—D and 6—D, cut off the perspective feet, Gk and kl, from G on GH, so that kk' and ll', parallel to AG, are the perspective of the traces of the front and rear faces of the pedestal. The intersections of the four traces now described form the perspective, Xt, of the base of the pedestal; vertical lines from the corners of which, are the perspective of its vertical edges. For the height, make Aa=3 ft., draw aE′, the perspective trace of the top plan of the pedestal on the neighboring wall; and draw the verticals, $k'k''$ and $l'l''$, which are the traces of the front and rear faces, upon the same wall. Then $k''m$, and l''Z, parallel to AG, will evidently limit the vertical edges, as at m, and Y. Also mZ and nY will, of course, vanish at E′.

3°.—*To find the perspective of steps ascending obliquely in the room.* To render the figure clearer, the size of the steps is considerably exaggerated. Let R, the foot of the foremost edge of the steps, be 3 ft. to the right of GH, and 6 ins. back of AG. Then make $Gx=6$ ins., draw the diagonal xD, and jR will be perspectively 6 ins. back of AG; and 5—E′ will cut off jR, perspectively equal to 3 ft. Let the plane of the sides of the steps intersect AK, at 8 ft. from A; then GD′ will cut off Ab, the perspective of 8 ft. on AK, and yRb will be the perspective horizontal trace of the plane of the side of the steps. Supposing them to be 7 ft. high, make $A7'=7$ ft., and 7′E′ will be the trace of the top tread upon the side wall AJ, and RT will be the perspective line containing the inner angles of the steps. This line, and therefore its projections, Rb and bT, are, in space, equally divided. Then, by Prob. LI., we find the true length of Rb. Thus, S′, the intersection of Rb with the horizon, is the vanishing point of Rb, and therefore the arc EF with centre S′, gives F, the vanishing point of chords making equal angles with AG and the original of Rb which is parallel to S′E (32). Remembering that such chords would cut off, from y, equal distances on yA, and on the original of Rb, we draw FRr and Fbf', giving rf' for the true length of Rb. Then divide rf', as desired, into equal parts, as at i, e, and g, and eF, etc., will divide Rb into the perspectives of these parts, as at i', e', and g'. Again, bT, being a parallel to the perspective plane, will be equally divided, if its original be so; hence divide bT, as at s,s', etc., into the same number of parts as rf contains. Now the verticals from i', etc.; and the horizontals from s, etc., and vanishing at S′ will intersect at i'', e'', etc., the inner angles of the steps, and at r'', o, etc.; the corresponding outer angles. Making ru, equal to 1 ft., and drawing uF, we find RU, the foot of the side board RTU; supposing RU to be, in reality, equal to 1 ft. Now UV and RT are parallel, and hence have the same vanishing point. The vanishing *line* of the plane (40) RbT is evidently a vertical line through S′, and the vanishing *point* of RT is in this line. Hence produce RT to meet S′S″, perpendicular to the horizon, and S″ (beyond the border) will be the desired vanishing point of RT and UV. The longer edges of the steps are perpendicular to Rb, hence ES, perpendicular to ES′, represents the visual ray parallel to them, and consequently gives S, the vanishing



68.

69.

71.

point of these edges, which are now simply drawn, from r'', i'', o, etc., towards S. In the figure, they are irregularly terminated, to avoid repeating constructions already explained.

The student can readily find the perspective of a given length on Rv; and the further edges, corresponding to $r''i''$, etc., would vanish at S′.

4°.—*To find a point of shadow, cast by the pedestal, on the side wall.* To avoid confusing the figure, this is the only shadow shown. After a little further progress, the student will find no trouble in constructing all the shadows afforded by this problem. Let H′ be the vanishing point of horizontal projections of rays, and R′ that of rays. Then XH′ is the perspective horizontal trace of a plane of rays containing Xn, also $x'x''$ is the trace of the same plane on the side wall, and x'', its intersection with the perspective ray nR′, is a point of shadow as required.

EXAMPLE.—*Let the light proceed from right to left, making H′ to the left of F, for example, and then find the shadows of the pedestal and of the steps.*

PROBLEM LXXIX.

To construct the perspective of a vertical regular hexagonal prism, and of its shadow upon the horizontal plane.

The perspective of the prism, Pl. VIII., Fig. 68, is here found by the method of perpendiculars and lines of the given object. That of the shadow, by the second general method (88) from the perspective of the object.

PROBLEM LXXX.

To find the perspective of a skeleton four armed cross at the centre of a square; together with that of its shadow on the horizontal plane.

1°.—*Preliminaries.* Let the square, Pl. VIII., Fig. 69, be in the horizontal plane with its edges oblique to the ground line. Such a selection of methods is here used, that no hori-

zontal projection is necessary. The horizontal visual plane, revolved about the horizon VH, into the perspective plane (Prob. XXV.) is used as a plane of projection for the point of sight, and the construction of vanishing points.

Now let *et* be the principal ground line, VH, as said, the horizon, E,E′ the point of sight, shown as just described, and D a vanishing point of diagonals, E′D being equal to EE′. Also suppose EV and EV′ to be parallel to the adjacent sides of the square, then, these lines being also visual rays, V and V′, their intersections with the horizon, are the vanishing points of the sides of the square. Next make VS=VE, and ES will be the visual ray parallel to chords cutting off equal distances on the ground line and on lines parallel to VE, reckoning from their intersection. S is therefore the vanishing point of such chords. Likewise S′ is the vanishing point of chords similarly related to lines parallel to EV′.

2°.—*The perspective of the square.* Let the distance of the centre of the square from the line EE′, be indicated at A, which thus becomes the intersection of the perpendicular, from that centre, with the ground line. AE′ is the perspective of this perpendicular. Let AB equal the true length of A*b*, or distance of the centre of the square, back of the ground line, and B will be the trace of a diagonal through the original of *b*. The perspective, BD, of this diagonal intersects AE′ at *b*, the perspective of the centre of the square. This done, V*bc* and V′*be* are the perspectives of the two centre lines, parallel to the sides of the square. Then, by drawing S*m*, we find *cm* equal to the original of *cb*, hence make *mo*=*mn*=half the side of the square, and draw *o*S and *n*S, which will meet *c*V at O and N, perspective middle points of two sides of the square. Likewise draw S′*bp*, giving *ep* for the real size of *eb*, then make *pq*=*pt*=*mn*, and *q*S′ and *t*S′ will meet *e*V′ at Q and T, middle points of the other two sides of the square, hence VQ and VT, V′O and V′N are the indefinite perspectives of the sides of the square, whose intersections, F, G K, and I, are the perspective corners of the square; of which F is before the perspective plane.

3°.—*The perspective of the cross and its shadow.* A vertical line at *b* is the indefinite perspective of the vertical arm of the cross. To find its summit, *d*, draw *m′c′*, the trace of the horizontal plane through that point, to indicate its true height, then project up, into *m′c′*, any point of the ground line from

which a line runs through b to a known vanishing point. Thus project e at c', and $c'V$ will limit bd at d, as required. Or, project p, A, and m, as shown, or B or e, and $p'S'$, $A'E'$, $m'S$, etc., will all locate the same point d. Suppose x, the intersection of the arms, to be at five-eighths of the height bd, then, as bd is vertical, bx equals five-eighths of bd; also the height of $B'c''$, the trace of the horizontal plane through x, will be five-eighths of the height of $m'c'$ above et. Hence project c at c'', and $c''V$ will contain x, and, supposing the horizontal arms to be equal and parallel to the sides of the square, project N at N′, and O at O′, on $c''V$, which completes the perspective of one arm. That of the other arm is similarly found, having V′ for its vanishing point.

The shadow is thus found. EH and E′R, being the projections of a visual ray of light, R and H are the vanishing points of rays, and of their horizontal projections, respectively. Then, for example, NH is the perspective of the horizontal projection of the ray through N′, and N′R is the perspective of the ray itself. Hence n' is the perspective of the shadow of N′. In the same manner, all other points of the shadow are found.

PROBLEM LXXXI.

To find the perspective of a skeleton pyramid, on a like pedestal; also the perspective of its shadow, on the pedestal and on the horizontal plane.

1°.—*The perspective of the pedestal.* In this problem, we pass from horizontal and vertical lines, to oblique ones, and their shadows, but, as will appear, without necessarily employing new constructions, though some will be used for further variety of illustration.

Let Nk, Pl. VIII., Fig. 71, be the ground line, not translated, $k'N'$ the trace, on the perspective plane, of the plane of the top of the pedestal, and CD the horizon, with C for the centre of the picture, D, for the vanishing point of diagonals, and therefore CD for the perpendicular distance of the eye from the paper at C.

$AD''m$ is a fragment of the plan of the pedestal, and M, of a corner of the base of the pyramid. Now, beginning with the method of diagonals and perpendiculars; $D''k—k'$ is the perpen-

dicular, and D″g—$k'g'$, the diagonal from the upper corner D″, k' of the pedestal. k'C is the perspective of the perpendicular, and g'D, of the diagonal, giving D′ for the perspective of D″. The vertical support, AA′, being in the perspective plane, there shows its full size, and A′D′ is an edge of the plate of the pedestal. A′D′, being produced to V, makes V the vanishing point of all lines parallel to AD″. Such a construction of V should, however, have a check, and so, let B′ and C′ be found, as D′ was, from their horizontal projections, not shown, but which are the other corners of the square, of which AD″ is one side. Then B′C′, produced, should contain V. Also A′B′ and D′C′, produced, should meet the horizon at the same point. The linear supports of the pedestal, being vertical, will be vertical in perspective (33—3°) and will be limited by the perspectives of either perpendiculars, diagonals, or lines parallel to AD″, through their feet. Thus P is found at the intersection of D′P, and gP, the perspective of the diagonal, D″g—kg, through D″,k, the projections of the foot of the support at D″. In like manner, B and C may be found.

2°.—*The perspective of the pyramid.* M, the horizontal projection of a corner of the base of the pyramid, is at the equal perpendicular distances, Mm and Mm'', from the sides of the square, and the other corners are similarly situated. Only the construction of one is therefore shown. The perpendicular, Mn—n', and the diagonal, MN—n'N′, from M, meet the perspective plane at n' and N′, respectively, in the trace of the upper plane of the pedestal, n'C and N′D are their perspectives, intersecting at M′, the perspective of M. The other corners, E,F, and G, may be similarly found; also EM and FG vanish at V. Finally, O being the centre of the square on AD″, and the plan of the vertex, make O′O″, drawn through O, equal to the height of the pyramid, and draw O″C, the perspective of the perpendicular, through the vertex. Then draw the diagonals FM and EG, not shown, of the base, and from their intersection, x, a vertical line will be the perspective of the axis of the pyramid, which will intersect O″C at the perspective vertex, V′.

3°.—*To find the shadows.* R, the vanishing point of rays, and H, that of their horizontal projections, are taken beyond the limits of the plate to favor a clear display of the shadow. Let it be remembered that H is in the horizon, and the line

HR, perpendicular to the ground line (Prob. LXXI.). Find, by a diagonal and perpendicular, the perspective of O,o, the centre of the upper base of the pedestal, and of O,O′, the centre of its horizontal projection. The former point is x, above mentioned; call the latter x'. Then xH will be the perspective of the projection of the ray through V′ upon the top of the pedestal. V′R is the perspective of the ray itself, and therefore v' is the perspective of the shadow of V′ on the plane of the top of the pedestal. Join v' with E, M, G, and F, by straight lines, and the portions of them, on the top of the pedestal will be the shadows of the edges of the pyramid upon that top. Recalling the point x', we have x'H, the perspective of the horizontal projection of the ray through V′, meeting V′R at v, the perspective of the shadow of the vertex on the horizontal plane. Next, a ray through f, where the shadow leaves the pedestal, will pierce the horizontal plane at f', the shadow of the lowest point of FV′, whose shadow falls on that plane. This point is found by dropping the perpendicular fe to meet BCV at e, and drawing eH, which meets fR at f', observing that eH is the perspective of the horizontal projection of fR. Vf is then the real portion of the shadow of V′F upon the horizontal plane. The shadows of B′ at b, of A′, at a, etc., are similarly found, by constructions like those of the two preceding problems, giving Aa for the shadow of AA′, etc.; and $abcd$ for the shadow of the top of the pedestal. The shadow of h, where the shadow M′h leaves the pedestal, is then at h', the intersection of the ray hR with cd, the shadow of C′D′. By finding $abcd$ at first, f' could have been likewise found as the intersection of fR with bc. Finally vh' is the perspective of the real portion of the shadow of M′V′ on the horizontal plane; and corresponding portions of the shadows of V′E and V′C might likewise be found. Only vh' and vf' would be real if the pyramid were solid.

Problem LXXXII.

To construct the perspective of the shadows cast on, and by, a vertical cylindrical scroll, the latter shadows falling on horizontal planes.

1°.—*The shadows on the horizontal planes.* The construction of the perspective of a vertical cylinder having been

already fully explained (Prob. XL.) the scroll $AfB-A'FB'$, Pl. VIII., Fig. 70, is assumed, with GL for the ground line, $G'L'$ for the trace, on the perspective plane, of any horizontal plane intersecting the scroll, and HH' for the horizon. Take H as the vanishing point of horizontal projections of rays, and suppose the three sections, AB, ab, and $A'B'$, of the scroll to be in parallel horizontal planes. The latter fact will appear by making C,c and C', the points of contact of the traces HC, Hc, and HC', of a tangent plane of rays on the three horizontal planes, all in the same vertical element CC'. Likewise, q, q', and Q, should be in the same vertical line, when Hq, Hq', and HQ are the traces of the other tangent plane upon the three horizontal planes. Now, taking R for the vanishing point of rays, the perspective ray, $C'R$, meeting its perspective horizontal projection, CH, at c'', gives Cc'' as the shadow of the element of shade, CC', on the horizontal plane containing AfB. Other points of the shadow of the scroll on this plane, may be found by the student, in the same way, as shown at F'', the shadow of F.

2°.—*To find the shadow of a secant, ST, of the upper base, upon the interior of the scroll.* First, find the horizontal projection, in perspective, of the secant, by projecting its intersections, as P, with the upper base, into the lower base, as at p, and join these points. Upon the line so found, project S, at s, and T at t, giving st for the perspective horizontal projection of ST. Now, to find the shadow of any point, as M, upon the scroll, project M at m, upon st, and mH will be the perspective of the horizontal trace of a vertical plane of rays through M. Such a plane will cut the scroll in an element, nN, which meets the ray MR at N, the shadow of M. The rest of the construction is left for the student to make.

Remark.—The construction of the shadows Bk, kK, and yN is the same as that of the shadow on the base and cylindrical part of a niche, cast by the front edge of the niche.

EXAMPLE.—*Repeat this problem with various relative positions of the rod and scroll, so as to show distinct shadows on the exterior of the scroll.*

PROBLEM LXXXIII.

To find the perspective of an arch-profile, and of its shadow, by the method of three planes.

N′e″, Pl. XI., Fig. 78, is the principal ground line, Oe′, in the back part of the horizontal plane, is the ground line of the perspective plane. The given object, being a simple one, is put behind the vertical plane, and principal ground line, so that no translation of the perspective plane is required, as in previous cases, before revolution, in order to make the right of the perspective, as at S, correspond with the right of the object, as F. E,E′ is the point of sight, and E′E″ the horizon. AF is the plan of the object, located in respect to the ground line, Oe′, of the perspective plane, and with reference to the eye at E,E′, looking towards that plane, and aiming to see, principally, the front of the object. The vertical projection of the arch is sufficiently indicated by the height of each point, on a line perpendicular to the ground line N′e″ through the plan of the same point.

1°.—*The perspective of the arch.* The edge, A—OA′, is in the perspective plane. Oe′ represents the vertical, as well as horizontal trace of the perspective plane, and, as indicating the former, it is the axis about which that plane revolves into the vertical plane of projection. In this revolution, A revolves to *a*, and *aa*′,=OA′, is the perspective of the vertical edge A—OA′. Next, B—B″B′ is the left hand edge of the arch. BE—B″E′ and BE—B′E′ are the visual rays from its extremities, and they pierce the perspective plane at *f*,*f*″, and *f*,*f*′, points which revolve in horizontal arcs, *ff*‴—*f*″*b*, and *ff*‴—*f*′*b*′, giving *bb*′ for the perspective of the vertical line B—B′B″. The perspectives of all the other points being found in the manner thus explained in detail, it is only necessary to explain the projections of the arch curves. Conceive the arch to be revolved to the left, about the vertical edge B—B″B′, till parallel to the vertical plane, and suppose it to be an equilateral pointed arch, that is, let the span, BD, and the chords of the arch curves form an equilateral triangle. BD—B′D′ will then appear at BD″—B′D‴. The line B′D‴ is the vertical projection of the revolved span; then, with radius B′D‴, and B′ and

D''' as centres, describe the arcs meeting at C''', and forming the revolved arch. Drawing the horizontal line $C'''C''$, and taking C midway between B and D, we have C,C'' for the summit of the arch, and C,C' for the peak of the whole body; points whose perspectives, found as before, are c'' and c'. For an intermediate point, take g''', project it at g'', counter revolve it to G,G' and find its perspective, g, as before. Continuing thus, the whole perspective may be completed.

2°.—*To find the shadow on the horizontal plane.* First: from the projections of the shadow. The ray of light must be taken obliquely downward from the left, in reference to the plane which the spectator is facing, that is to the perspective plane, Oe'. Hence $AM-A'M'$ is such a ray. It pierces the horizontal plane at M, the shadow of A,A'. Then, as for all other points thus far found, $ME-M'E'$ is the visual ray from the point of shadow M,M' (M being also the point itself). This ray pierces the perspective plane at a point whose revolved position is m, found as were other points, before. In like manner, all the points of the shadow might be found, but having revolved the perspective plane, with the complete perspective figure, into the vertical plane of projection, we can recur to the usual method with two planes, thus: Second: to find the required shadow by vanishing points. The perpendicular visual ray meets the perspective plane at the point whose horizontal projection is e. Then make $Oe''=Oe$, and draw the perpendicular to OOe'' through e''; and make $e''E'''=Ee$, and project E''' at E'' on the horizon, then E''',E'' will be the revolved point of sight. Knowing the direction of the light, the vanishing points of rays and of their horizontal projections can be found as in previous cases; observing, that, as a revolution of 90° has taken place, the horizontal projection of the ray will be perpendicular to AM. In case the projections of the light make the usual angle of 45° with the ground line, it is obviously only necessary to make $E''H=HR=Ee$, to obtain R and H, the vanishing points of rays and of their horizontal projections. Having found these points, the shadow of any point, as c', is at the intersection, o, of the perspective ray $c'oR$, and its perspective horizontal projection cH.

Problem LXXXIV.

To find the perspective of a square pillar and abacus, and of the shadow of the abacus upon the pillar.

1°.—*The perspective of the object.* Let ABCD, Pl. XI, Fig. 82, be the plan of the abacus, after revolving the horizontal plane 180° (Prob. XXIII.) and let the face AB of the abacus be in the perspective plane. This face will then be its own perspective, as shown at A′B′*ab*. The concentric square, FGHK, is the plan of the pillar, E, the centre of the picture; V, the vanishing point of diagonals; R, that of rays, and P, that of their horizontal projections.

Draw the diagonal CB. Its perspective, BV, intersects G′E and F′E the perspectives of the perpendiculars HG′ and KF′, at *g* and *k*, the perspectives of G and K. Then, as FG and HK are parallel to the ground line, their perspectives will be so also, hence *gf* and *kh*, thus drawn, will determine *f* and *h*, the remaining corners of the base of the pillar.

Perpendiculars to the ground line, at *f*, *g*, *h*, and *k*, will be the perspectives of the vertical edges of the pillar. They are limited by the perpendiculars through their uppermost points. Thus, project F′ at F″, since the lower base of the abacus is horizontal, and F″E will be the perspective perpendicular which limits the perspective vertical edge, *ff′*, at *f′*. Then *f′g′*, parallel to the ground line, limits *gg′* at *g′*, and *g′*E limits *hh′* at *h′*, as *g′*E is the perspective of the upper perpendicular, horizontally projected in HG′.

Returning to the abacus, the diagonal BV meets the perpendicular AE at *c*, the perspective of the horizontal projection of the point C,A′. Then the vertical line *cc′* meets A′E, the perspective of the lower left edge of the abacus at *c′*, the perspective of the under corner, C,A′. Then *c′*D′, the perspective of the lower edge at CD, limits B′E at D′ and the short vertical line D′*d* limits *b*E, which completes the perspective of the abacus.

2°.—*To find the perspective of the shadow of the abacus upon the pillar.* The edge AB—A′B′ casts this shadow, and as this edge is parallel to the face of the pillar, its shadow will be parallel to the edge itself. Also as both are parallel to the

perspective plane, the perspective of the shadow will be parallel to the shadow itself. Hence it will only be necessary to find one point of it. Any line as PQ, intersecting the *perspective of the horizontal projection of* AB—A′B′, that is, in this case, AB itself, and *fg*, is the perspective of the horizontal trace of a vertical plane of rays, which cuts from the edge AB—A′B′ a point, Q′, and from the face of the pillar, the vertical line *qq′*. The shadow of Q′ falls on *qq′* at *q′*, found by drawing the perspective, Q′R, of the ray of light through Q′. Then *pq′n*, parallel to the ground line, is the required shadow of the abacus upon the pillar.

Remarks.—a. A ray R*n*, not shown, would intersect A′B′ at the extreme right hand point on that edge, which would cast a shadow, *n*, on the pillar. The shadow of the remainder of that edge to the right, falls on the horizontal plane.

b. Likewise, a ray R*p*, not shown, determines the last point of B′A′, towards the left, whose shadow *p*, falls on the front of the pillar. The remainder of this edge casts a shadow on the left-hand face of the pillar.

c. If EP were greater than EV, it would be apparent, on making the construction, that a portion of the shadow of A′*c′* would fall on the front of the pillar.

d. Finally, if E were at the left of the pillar, the left side of the pillar and the shadows upon it, would be visible. The student should, for practice, make the easy constructions indicated in these remarks.

Problem LXXXV.

To find the perspective of a cylinder, which is oblique to both planes of projection, and of its shadow on its own interior, and on the horizontal plane.

1°.—*Preliminaries.* Let the cylinder, Pl. IX., Fig. 72, be inclined at the usual conventional angle adopted for the direction of light, that is, let its axis make an angle of 35° 16′ with each plane of projection. Then let OO″, making an angle of 45° with the ground line ZD′, be the horizontal projection of this axis. Take KN′, parallel to OO″, for the ground line of an auxiliary vertical plane, parallel to the axis, and make a

right angled triangle C′O‴C″, such that O‴C″ shall be equal to the hypothenuse of an isosceles right angled triangle, whose other sides, each, equal C″C′. Then C′O‴ will make the given angle of 35° 16′ with KN′, and will be the auxiliary projection of the axis of the cylinder. Let the cylinder be limited by a circular right section, G′Q‴, for its upper base, and by its elliptical horizontal trace, N′Q‴, for its lower base. The horizontal projection can now be completed by elementary operations not necessary to explain here.

The ground line ZD′, being secant to the horizontal projection, shows that the perspective plane cuts the cylinder. But, remembering always that the perspective plane, being an imaginary surface, in nowise affects the *apparent figure* of a *given object*, seen from a *given point*, the required perspective can be found as usual, only the perspective of the upper base will be larger than the base itself, it being, in this case, a section of the visual cone, made by a plane, further from the eye, than that base of the cylinder. This being understood, let Z′N″ be the ground line of the translated position of the perspective plane, E′D the new, and E″S′ the original projection of the horizon. E,E″ the real, and E,E′ the translated position of the eye. Make E′D=E*e*″, and D will be the vanishing point of diagonals. Furthermore, the elements of the cylinder, being parallel, will have a vanishing point, hence draw ED′, and E″V′, at angles of 45° with the ground line, and make V″V (V not shown) perpendicular to the ground line, and equal to D′V′, and V will be the vanishing point of the elements of the cylinder, useful as a check upon the points, separately constructed, of the two bases.

2°.—*The general perspective of the cylinder.* Having completed the foregoing preliminaries, we proceed to the perspective of the cylinder, and first will determine the extreme visible elements of its convex surface. To do this, find the elements of contact of two tangent visual planes. These planes will contain, in common, the visual ray EH—E″H′, parallel to the axis of the cylinder, hence their horizontal traces, HP and HJ, will contain H, the intersection of this ray with the horizontal plane, and will be tangent to the base, MNQ″, of the cylinder.

The elements, as PB, through P and J, will be the extreme visible elements, whose perspectives are the straight visible limits of the perspective of the cylinder, beginning near *q*′ and

T. Points in the perspectives of the bases may be found by any convenient method already explained. That of diagonals and perpendiculars is mostly used in the figure. Thus, the diagonal $Q''s$, and perpendicular $Q''i$, are in the horizontal plane, hence their perspectives are $s'D$ and $i'E'$, intersecting at q', the perspective of $Q''Q'''$. Again, taking a point B, on the upper base, and in front of the perspective plane, we first draw the trace, op', of the plane of its diagonal and perpendicular, at a height above $Z'N''$ equal $b'B'$. Then the diagonal, Bp, and the perpendicular from B, meet the perspective plane at p' and o, and Dp' and $E'o'$ are their perspectives, which, being produced, intersect at b, the perspective of B. Other points may be similarly found. Points very nearly in the plane of the horizon, as c, can best be found by vertical visual planes and perpendiculars. Thus Ec' is the horizontal, and $c'c$ the vertical trace of such a plane, through C, and $E'q$ is the perspective of the perpendicular from the same point, q being projected from C, and at a height above $Z'N''$ equal to $C'C''$. Then cc' meets $E'q$, produced, at c, the perspective of C.

By finding, first, all the points, as u, of the upper base, the corresponding points, as T, of the lower base will be at the intersection of either a diagonal, or perpendicular, or trace of a visual plane, with the elements of the cylinder, as uV, drawn through the former points. Finally, the perspectives of tangents to the lower base and parallel to the ground line, will themselves be likewise parallel; tangent perpendiculars will meet at E', in perspective; and the vertical traces of vertical visual planes, tangent to that base, will be tangent to the perspective of the base, and will be perpendicular to the ground line.

3°.—*To find those points of the upper base, whose perspectives are the highest and lowest.* These points will be the points of contact of the traces of two visual planes, upon the plane of the upper base, and containing a parallel to the ground line, through the eye. These traces will, in perspective, be horizontal tangents (Theor. IX.) to the perspective of the upper base. Now $G'K$ is the auxiliary vertical trace of the plane of the upper base, and ZK, perpendicular to the ground line KN', is its horizontal trace, since the auxiliary plane of projection is made perpendicular to the plane of the upper base. Z is a point in the principal vertical trace of the same plane, and by making Le' equal to Le, we shall have the two vertical projec-

tions of the vertical line at L, in which the auxiliary and primitive vertical planes intersect, hence Ze' is the primitive vertical trace of the plane of the upper base. We have now only to find the intersection of the horizontal line, ES—E″S′, with this plane, and to draw from this point tangents to the upper base. A vertical plane through ES—E″S′ meets the plane $e'Zt$ in the line tS—t'S′, parallel to Ze', hence ES—E″S′ meets the latter plane in S,S′, from which the tangents, of which SF is one, determine the points, as F, whose perspectives, as at f, are the highest and lowest.

Remark.—It thus appears that the highest and lowest points of the perspective of any section of the cylinder are not on its elements of apparent contour. Indeed the highest points, for example, of different non-parallel sections, will not be on the same element of the cylinder. Thus, the ellipse, I′$r'f''$, is the section of the cylinder, contained in the perspective plane, and is therefore its own perspective, and II′, the point of contact of a horizontal tangent, is its highest point, which is seen to be on a different element from the one through F, or h'' the remotest limit of the lower base. The highest, or the lowest, points of two parallel sections, as G′Q′ and G‴Q‴, will, however, evidently be on the same element.

Finally, should it be desired to find the traces of the visual planes, tangent to the cylinder, upon the plane of the upper base, it can be done as follows. d'H′H″ is a visual plane, perpendicular to the vertical plane, and H″d, its intersection with the plane, d'ZH″, of the upper base, meets the visual ray EH, common to the two tangent planes, at d'' in horizontal projection. Then tangents from d'' will have B, on the element from P, and the upper point of the element through J, for their points of contact with the upper base.

4°.—*To find the perspective shades and shadows of the cylinder.* And, first, the vanishing points of rays, and of the horizontal traces of planes of rays containing elements of the convex surface. Assuming ER″—E″R′ for the direction of the light, R′, its intersection with the perspective plane before translation, is the vanishing point of rays, appearing, after translation, at R, below Z′N″, a distance equal to R′R″. The ray of light, OX—O′X′, pierces the horizontal plane at the point denoted by X, and XO″ is therefore the horizontal trace of a plane of rays through the axis; a plane to which those

containing elements are parallel. But EU′—E′U is the visual ray parallel to XO″, and it meets the perspective plane in the horizon (33—2°) at U, which is therefore the vanishing point of the required horizontal traces. Then UT and UT′ are the perspective horizontal traces of two tangent planes of rays, and T*u* and T′*x*, which vanish at V, are the perspectives of the elements of shade, whose shadows are the shadows of the convex surface of the cylinder. These shadows are the traces TU and T′U, limited at *x*′ and *u*′ by the rays *x*R and *u*R. Any secant plane, as UY, cuts two elements, Y*y* and W*w*, from the cylinder, giving W*w*′, limited by the ray *w*R, for the shadow of W*w*. Also if the ray *y*R intersects W*w*, as in this case it happens to just at W, the latter point is a point of shadow cast on the interior of the surface, if hollow, by the opposite point, *y*, of the upper base.

THEOREM XXII.

The perspectives of the elements of apparent contour of a cylinder, will be tangent to the perspectives of its plane sections; but will be so at the extremities of an axis, only when the axis of the cylinder is parallel to the perspective plane, and in a visual plane perpendicular to the perspective plane, and when the planes of the sections are perpendicular to the latter plane.

Figure 13 will assist the imagination in representing to itself the statement of the following demonstration.

Conceive two visual planes, tangent to such a cylinder along its elements of apparent contour, E and E′. These elements will be in a plane parallel to the perspective plane, PQ. Hence when the visual plane containing the axis, is situated as described, any plane, perpendicular to it, and cutting the cylinder, will cut from the plane of the elements of contact E and E′, a chord, C, of the section, which will be parallel to the perspective plane, and perpendicular to the elements E and E′.

But the chord and these elements, all being parallel to the perspective plane, their perspectives will be parallel to the lines themselves, and will consequently have the same relative positions as those lines. Now the parallel perspectives, *e* and *e*′,

of E and E′ bound the perspective of the convex surface; *and hence are tangent to the perspectives of all its curved sections*, and the perspective of C is perpendicular to *e* and *e*′, and is a chord *c* of the elliptical perspective, *s*, of the section S. But when the tangents at the extremities of this chord, *c*, of an ellipse, are parallel to *each other* and perpendicular to *it*, that chord is no other than an axis of the ellipse *s*.

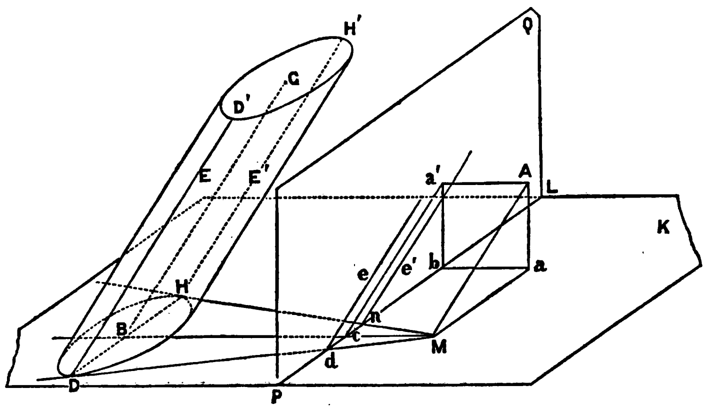


FIG. 13.

Finally, 1. If the *axis of the cylinder* were oblique to the perspective plane, the traces of the tangent visual planes, being no longer parallel, would not be parallel tangents to the perspectives of sections of the cylinder.

2. If the plane of the section, S, was not situated as described, the chord C would not be perpendicular to E and E′, and hence its perspective, *c*, being parallel to C would only be a diameter, but not an axis of *s*. And—

3. If the axis of the cylinder were not in the perpendicular visual plane described, the plane of E and E′ would no longer be parallel to the perspective plane; hence the perspectives of the chords, C, of plane sections of the cylinder would not as before be parallel to the chords themselves, and so, not perpendicular to *e*, and *e*′.

But the three last suppositions embrace all the distinct changes possible in the conditions of the theorem. Hence the theorem is true, as stated, for those conditions only.

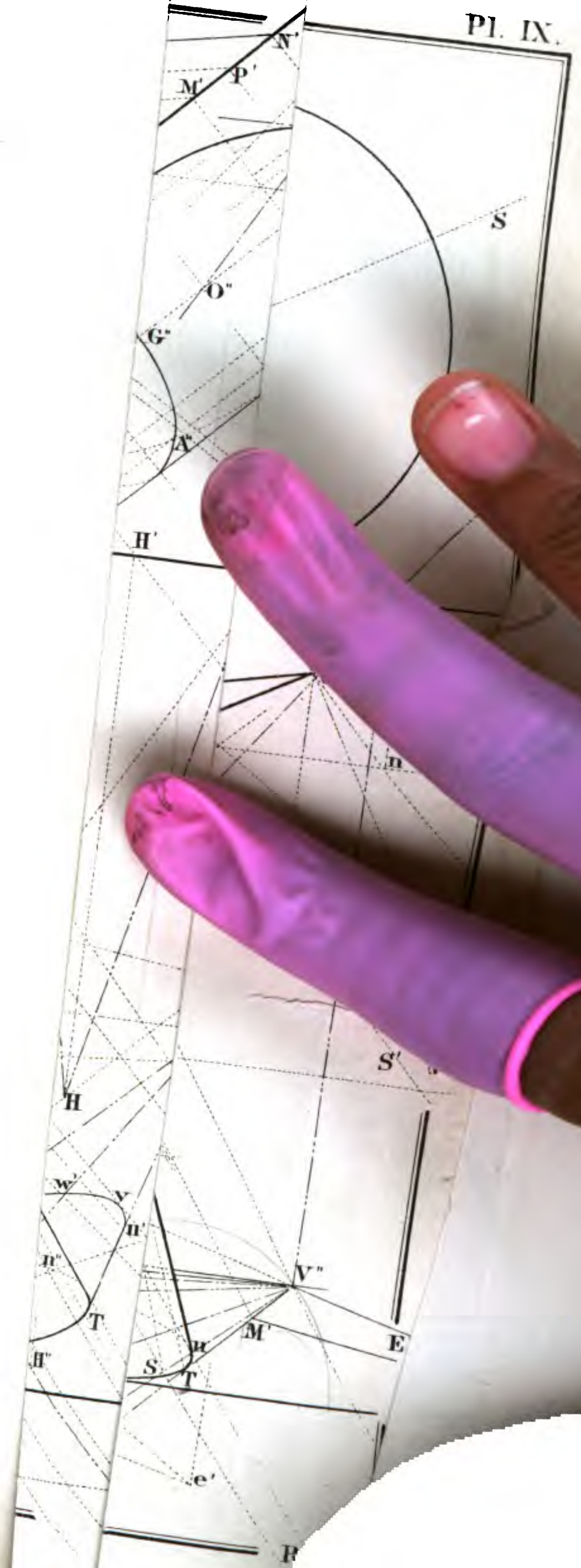
PROBLEM LXXXVI.

To find the perspective of a cone, bearing a horizontal quarter square at its vertex; of the shadow of the square on the cone, of the cone on itself, and of both on the horizontal plane.

1°.—*To find the perspective of the cone and plate.* We will again employ, chiefly, the method of diagonals and perpendiculars, with the perspective plane translated forward before revolution. The circle with radius VA, Pl. IX., Fig. 73, is the horizontal projection of the cone, and FVE, that of the quarter square, of which the part AEm' is in front of the perspective plane. J$'$A$'$ is the translated ground line; C$'$, the centre of the picture; D, beyond the plate, the vanishing point of diagonals, and C, the station point C$'$C being equal to C$'$D. Also V$'$ is the vertical projection of the vertex, and V$'$F$'$ that of the quarter square. The perspective of the base requires no unusual operations to construct it. The determination of only one of its points, *b*, is therefore shown. B is the original of this point, the perspective of the perpendicular from which is B$'$C$'$, and that of the diagonal, Bm', is m''D, meeting B$'$C$'$ at *b*. The perpendicular from the vertex meets the perspective plane at V$'$, and the diagonal, Vm', at the point between *k* and *l*. The perspective, V$'$C$'$, of the perpendicular meets that of the diagonal at *v*, the perspective, therefore, of the vertex VV$'$. The line VF, being parallel to the ground line, its perspective, *vf*, is parallel to it also, and is limited by F$'$C$'$, the perspective of the perpendicular from F. The line VE, itself, is the perpendicular from E, and its perspective is drawn through C$'$ and *v*, and is limited at *e* by *en*$'$D, the perspective of the diagonal E*n*—V$'$*n*$'$. Thus *evf* is the perspective of the quarter square.

2°.—*To find the intersection of the ray through V, V$'$, with the horizontal plane and with the vertical plane through EF.* The ray, VS—V$'$S$'$, pierces the horizontal plane at a point readily found, and indicated by S. Then SPV$''$, is a vertical visual plane, from SS$'$, whose horizontal trace, SP, contains C, and whose vertical trace, PV$''$, meets the perspective perpendicular S$'$C$'$, at V$''$, which is the perspective of the intersection, S,S,$'$ of a ray of light through the vertex, with the horizontal plane, that is, V$''$ is the perspective of the shadow of V,V$'$ on

Pl. IX.



that plane. Again; the ray SV—S′V′ meets the vertical plane through FE in the point NN′, whose perspective, here found by a diagonal and perpendicular, is V‴. This done, all lines through V″ are perspective horizontal traces of planes of rays through the cone's vertex, *v;* and all lines through V‴ are traces of the same planes on the vertical plane through *ef.*

3°.—*To find the shadow of EF upon the cone.* In this construction, we may either find a point of shadow of *some* point of EF upon a *given element* of the cone; or of a *given point* of EF upon *some* element of the cone; or the point of shadow determined by a *given plane of rays.*

The tangents, V″*q*G, and V″TM, are the perspective horizontal traces of tangent planes of rays, and therefore determine the elements of shade, *qv*, and T*v*, of the cone. Then, to proceed, find *c*, the perspective of the centre, V,A′, of the cone's base, then *cf*′, equal and parallel to *vf*, is the perspective horizontal projection of *vf*, that is, of VF. Also *ve*, being, in space, the perpendicular, VF—V′, we have *c*A′, limited at *e*′ by the line *ee*′, perpendicular to the ground line, for the perspective horizontal projection of *ve*. Hence *e*′*f*′ is the perspective horizontal projection of *ef*, which, being, here, a diagonal, *e*′*f*′ would vanish on the horizon, to the left of C′ by the distance CD=CC′. Now drawing V‴G and V‴M, we have the perspectives of the traces, on the vertical plane through *ef*, of the planes of rays tangent to the cone, therefore *gm*, the portion of *ef* between these traces, is the portion whose shadow falls on the cone. The elements of contact, *qv* and T*v*, of these tangent planes are met by the rays *g*R and *m*R, at *g*′ and *m*′, the extreme real points of shadow. Similarly, the point of shadow is found on any particular assumed element. Thus, to find the shadow on the extreme element *rv*, through *r* draw V″*rf*′ for the perspective horizontal trace of the plane of rays containing this element, *f*′V‴ is the trace on the plane V‴GM, and *o* is the point cut from *ef* by this plane. The ray *o*R then cuts the element *rv*, at *o*′ the required point of shadow, and, in this case, the visible limit of the shadow. Again, to construct the shadow of a given point as *h*, draw the vertical trace V‴H of a plane of rays through *h*, and HV″, the horizontal trace of the same plane, will determine the element *dv*, which is met by the ray *h*R at *h*′, the shadow of *h*. Lastly, assuming the traces as V″K and V‴K of any plane of rays through *v*, we determine

by the former trace the element $A'v$, and by the latter the point k, then the ray kR intersects $A'v$ at k', a point of shadow.

4°.—*The shadow of the cone on itself, and of the cone and quarter square of the horizontal plane.* These can now both be found in a moment. Suppose the quarter square, and the part of the cone above the shadow just found, to be removed, then the portion, $g'h'm'$, of the upper edge of the remaining frustum, will cast a shadow on the opposite interior surface. Thus, any plane of rays, as $V''H$, cuts the frustum in the opposite elements, dh' and pv, and h', the uppermost point of the former, casts a shadow on the latter at h'', its intersection with the ray h'R. As before, we can begin with a *given element*, to receive a point of shadow; a *given point* casting a shadow; or a *given plane* of rays, containing both a point of shade and an element. This shadow of the cone on itself begins at g' and m', on the elements of shade, which, being in tangent rays, are where points casting shadows unite with their own shadows.

Again: produce the ray from m to the trace $V''M$ of the plane of rays through it, and M' will be the shadow of m on the horizontal plane. Now EV, being a perpendicular, its shadow is one also, hence $C'V''$, produced, and limited by the ray eR, gives $V''E''$, the shadow of VE; and $M'E''$, the shadow of me. Also $V''F''$, parallel to vf, and limited by the ray fR, is the shadow of vf; then the portion of $M'F''$ outside of $V''q$, is the remainder of the shadow of fe on the horizontal plane; and $TV''q$ is the shadow of the cone on that plane. $V''F''$ and $V''E''$, by being wholly exterior to the latter shadow, indicate that vf and ve cast no shadow at all on the cone.

Remark.—The extent of the visible part of a conic surface of revolution, depends on the relative distance of its vertex, and the eye, from the plane perpendicular to its axis. Thus in Fig. 73, if the eye were at the *same height as the vertex*, just half the cone would be seen; if it were *above*, more and more, and, finally, the whole, would be visible, and if it were *below*, less would be seen.

This statement can be readily modified in any given example to suit the case of any oblique cone.

Problem LXXXVII.

To find the oblique perspective of a monumental arch.

Let the perspective plane, Pl. X., Fig. 78, be translated backward, from its first position, at FG, to the new position F′G′; let ADBC be the plan of the arch, and *abcd*, of the archway. E and E′ are the projections of the point of sight upon the perspective and horizon planes; E″ is its projection upon the horizontal plane of projection. EV and EV_1, drawn parallel to AB and AD, determine V and V_1, the vanishing points of the latter lines and of all parallels to them. The arc EV_2, being drawn with V as a centre, the chord EV_2 is the visual ray parallel to the chord, B*m*, of the arc B*m*, whose centre is A; hence V_2 is the vanishing point of B*m*, and of all parallels to it.

1°.—*To find the perspectives of all the straight edges of the arch.* The vertical edge at A, being in the perspective plane, is its own perspective, shown at A′A″, which, in practice, would be laid down in its proper size, according to the scale of the plan. A′V and $A'V_1$ are then the indefinite perspectives of AB and AD; they are limited, the former at B′ by $m'V_2$, the perspective of the chord B*m*, and the latter at D′, by F′V, the perspective of CDF. Then $B'V_1$ and F′V meet at C′ the remaining external angle of the base. Drawing the arcs *aq* and *bn*, with A as a centre, and translating *q* and *n* to *q′* and *n′*, we find $q'V_2$ and $n'V_2$, which intersect A′B′ at *a′* and *b′*, the perspectives of *a* and *b*. Otherwise, as shown at *a′*, the perspective, *r*E′, of the perpendicular *ar*, meets A′B′, or $q'V_2$, at *a′*. And once more, as shown at *b′*, *cb* is produced to G, which is translated to G′, whence $G'V_1$ is the indefinite perspective of *cb*, and determines both *b′* and *c′*. The point *d′* may be similarly found. At the seven points now found, besides A′, erect vertical lines, and limit D′D″ by $A''V_1$; B′B″, by $A''V_1$, and C′C″, if shown, by D″V or $B''V_1$. Let A′*o′* be the height of the vertical part of the archway, then *o′*V limits the verticals, *a′a″* and *b′b″*, at *a″* and *b″*, it being the perspective of a line in the face of the arch and parallel to the base line AB. Finally $a''V_1$ and $b''V_1$ limit the verticals, *c′c″* and *d′d″*, at *c″* and *d″*; which step completes this branch of the problem.

2°.—*To find the general perspectives of the arch curves.* Conceive the face of the arch to be revolved about $A-A'A''$ as an axis, and into the perspective plane. The centre, e, of the front arch line, then describes the arc eo, in the horizontal plane $o'M$, giving O', projected up from o, for the revolved centre, and the semicircle MLN, with radius equal to ae, for the revolved face line of the arch. Now let L be any point of the face line. It is also evidently the intersection of a chord parallel to Bm, from the original of L, with the perspective plane, hence LV_2 is the perspective of this chord. Also, drawing Ll, parallel to the ground line, lV is the perspective of the original of Ll, and L', the intersection of lV and LV_2, is the perspective of L. The corresponding point in the rear face line, cd, is on LV_2 and may readily be found, as shown for the point d'', by a construction similar to $o'o''V_1$ and $o''d''V$.

3°.—*To find particular points in the perspectives of the face lines.*

1. To find the perspective of the *highest point*, K, whose horizontal projection is e. Drawing the horizontal tangent KK' (not shown), the point K' is the intersection of this tangent with the perspective plane, and is, therefore, one point of its perspective, $K'V$. The perspective of K, would be the intersection of $K'V$ with the perspective KV_2, of a chord in the system parallel to Bm; or with the perspective (not shown) of the perpendicular at e, considered as in the horizontal plane KK'. But again, the point is thus found in the figure: translating e to e', we have $e'E'$ the perspective of the perpendicular ee'; and e'', its intersection with $A'B'$, the perspective of e; then $e''V_1$ determines the corresponding middle point, g, of the rear base line $c'd'$. Now a vertical line, $e''e'''$, is the centre line of the front of the arch, and gg' that of its rear; the former meets $K'V$ at e''', the perspective of the highest point of the front curve; the latter meets kV—found by first drawing $K'V_1$ at g', the perspective of the highest point on the back curve.

2. To find the point at which the *perspective tangent is horizontal.* This will be the point of contact of a plane containing a visual ray, $EI-E'I'$, parallel to the ground line. The trace of this plane on the face of the arch, will contain the point where $EI-E'I'$ pierces that face, which is I; since $A'''I$ is the trace of that face on the horizontal plane containing EE'. Revolving the plane of the face about $A-A'A''$ as before, I

appears at I′, and I′T is the revolved trace required, giving T as the revolved original of the desired point. The point T″ of this trace is fixed and is one point of the trace, T″T′, which is horizontal, on the perspective plane. T′, the perspective of T, is now at the intersection of T″T′ with *t*V the perspective of T*t*, a horizontal line in the front face. The corresponding rear tangent can now be immediately found.

3. To find the *highest visible element* of the cylindrical surface of the arch, supposing it to be visible. This line is the element of contact of a visual plane, perpendicular to the face of the arch, and therefore containing the visual ray EE_1 whose revolved position is E_2 in vertical projection; and which is perpendicular to the plane, A‴I, of that face. E_2J is the revolved trace of this visual plane, and J is the revolved position of the front end of the element sought. In the counter revolution of the arch face about A—A′A″, the point J, whose horizontal projection is *p*, returns to *ff′*; and the perspective perpendicular, *f′*E′, meets the perspective, *j*V, of the horizontal through J, at J′, the perspective of J, so that $J'V_1$ is the perspective of the element required, and is tangent to both the front and rear arch curves. Having now, in all, five tangents to each arch curve, with the point of contact of each, besides the point L′, the curve can be accurately sketched.

4°.—*To find the shadow of the archway upon itself.* This, to avoid too much complication of lines, is the only shadow here found. Let H be the vanishing point of horizontal projections of rays and R′, that of rays themselves. First, *a′*H is the perspective horizontal trace of a vertical plane of rays through *a′a″*, and the vertical line at U is its trace on the opposite side of the archway, and is the indefinite shadow of *a′a″*. This shadow is limited at U′ by the ray whose perspective is *a″*R′. Above U′ the shadow is cast by the curve *a″*J′*b″*; and its limit will be at the element of contact of a plane of rays tangent to the cylinder of the arch. The trace of this plane on the arch face will contain the projection of R′R upon the face AB of the arch, since the plane of rays is perpendicular to that face. R′ whose horizontal projection is R, is projected on the plane AB, at PP′, found by drawing the horizontal RP, whose vertical projection is R′P′, perpendicular to AB. The vanishing point of the perspective of this trace will be the intersection of a visual ray parallel to it, with

the perspective plane. E″S, parallel to AB, and E′R, parallel to E″″P′—found by projecting E″E′ upon the plane AB at E″E″″—are the projections of this ray, which therefore pierces the perspective plane at $R_{,}$, the required vanishing point. QR_1 is now the perspective of the trace described, and Q is the extremity of the shadow sought. Any secant, as $e'''R_{,}$, is the trace of a plane cutting a point, e''', from the arch line, and an element $L'V_1$ (L′ being here confounded with the perspective of L) from the cylindrical surface. Then S, the intersection of this element with the ray $e'''R'$, lying in the same plane of rays with $L'V_{,}$, is the shadow of e'''. Any other intermediate points could be found in like manner.

EXAMPLE.—*Find the perspective of the frustrum of a quadrangular pyramid, with a projecting circular tablet on each face; and the perspectives of its shadows.*

PROBLEM LXXXVIII.

To find the perspective of the shadow cast by the vertical great circle, parallel to the perspective plane, of a hemisphere, upon the visible interior of the surface.

This problem is an extension of that topic in the problem of the niche, which treats of the shadow on its spherical part.

Let the vertical great circle FG—F′CG′L, Pl. XI., Fig. 81, coincide with the perspective plane; let E be the centre of the picture, D and D′ the vanishing points of diagonals, and DD′ the horizon. The points of shadow may be classed as general points, and particular points, the latter being located on preassigned circles of the sphere. The former points may be very conveniently found from their projections.

1°.—TO FIND POINTS, IN GENERAL, OF THE SHADOW.

1. *By the method of projections of the point.* Let ER be the vertical projection of the visual ray of light, and R the vanishing point of rays. Then aa', for example, parallel to ER, is the vertical trace of a plane of rays, cutting the hemisphere in a small semicircle, whose vertical projection is the chord aa'. Now if this plane be revolved to the left into the vertical, i. e. the perspective plane, the semicircle, being behind the latter plane, will appear to the right of the axis aa', as at aba'. The parallel plane, with ER for its trace, being similarly

revolved, E, being in front of the perspective plane at the distance ED′, will appear at E′ by making EE′ perpendicular to ER, and equal to ED′. Then E′R is a ray of light, when revolved into the perspective plane. Hence *ab*, parallel to E′R, is the revolved position of a ray in the plane *aa*′, and *b*, that of a point of shadow. In counter revolution *b* returns in an arc whose projection, *bb*′, is perpendicular to *aa*′, and is found at *b*′. The line *bb*′ is also the revolved position of the perpendicular from the point of shadow, and which pierces the perspective plane at *b*′. Now *b*′E, the perspective of this perpendicular, contains the perspective of *b*′, and so does *a*R, the perspective of the ray through *a*, the point whose shadow is *b*′, hence *b*″, the intersection of *b*′E and *a*R, is the perspective of *b*′.

2. *By auxiliary shadows.* Another method for finding any point of the shadow, is by an application to perspective constructions of the beautiful method of single *auxiliary shadows.* Thus, the shadow of the vertical front circle, upon any plane parallel to it, will be a circle, and the intersection of this shadow with the circle cut by the parallel plane from the sphere, will be a point of shadow on the spherical surface. To find the perspective of the latter circle first find that of the horizontal semicircle FKG—F′G′. This is conveniently done by diagonals and perpendiculars. Thus, O′E is the perspective of the perpendicular from K,O′, and F′D, that of the diagonal, giving F″, the perspective of K,O′. Again, *k*′E is the perspective of the perpendicular from any point, *kk*′, whose horizontal projection, and diagonal, by the method of (Table IV.—3°) would be *k*″, and *k*″*q*. The perspective of *k*″*q* is *q*′D, which meets *k*′E at *k*‴, the perspective of *kk*′. Other points being found in the same manner, and joined, F′F″*k*‴G′ is the perspective of the given semicircle. Now assume any chord, *rk*‴, parallel to F′G′, and on it as a diameter, describe a circle which will be one, cut from the spherical surface by a plane parallel to the front circle. The perspective of the shadow of F′O′, on this plane, will be the radius of the perspective of the shadow of the front circle on the same plane, then draw F′R and O′R, and, observing that *s* is the perspective of the projection of O′ on the parallel plane, draw *su*′, parallel to ER, and it will be the perspective of the projection of O′R on the parallel plane; since this projection is parallel to the perspective plane, and

therefore to its own perspective. su' meets O′R in u', the perspective of the shadow of O′, hence $u'u$, parallel to F′O′ and limited by F′R, is the perspective radius of the shadow UuI, which meets $k'''u$I, contained in the same parallel plane, at U and I, two points of the required shadow.

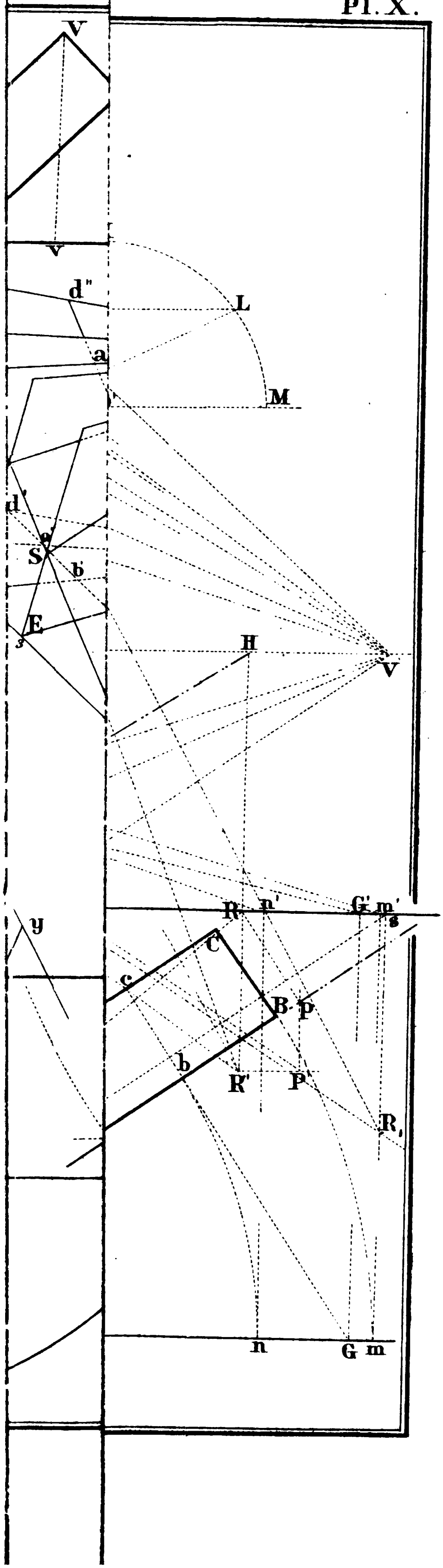
2°.—PARTICULAR POINTS.

1. *In given planes of rays.* Among these are the following. The shadow on the great semicircle cc', which is in a plane of rays. This point, whose revolved position is d, vertical projection, d', and perspective, d'', presents no peculiarities except that the revolved position of cc', it being a great circle, coincides with cAc'. We pass therefore to the shadow on the small circle projected in ee', and which is contained in the visual plane of rays of light, eER. Here the perspectives, f'E, of the perpendicular, revolved at $f'f''$; and eR, that of the ray from e, the point casting the shadow, coincide, and some other auxiliary line must be used. But the shadow, f', is behind the perspective plane at the distance $f'f''$, as seen at f, in the plan, hence a diagonal through it would pierce the perspective plane at g, found on the horizontal line $f'g$, at the distance $f'f''$ from f'. The perspective, gD, of this diagonal, meets f'E at f''', the perspective of ff'.

2. *Other particular points. First;* the point on the vertical great circumference F′AL. These are merely the points of contact, A and B, of vertical projections of rays tangent to the sphere. AB is the common diameter of the two great circles, F′AL and the circle of shadow, cut from the sphere by the cylinder of rays whose given section, or base, is F′AL, and all of whose elements are secants of the sphere save the two which are tangents at A and B.

Second; to find the points on the horizontal great semicircle FKG—F′G′. We begin *a first method* by finding the projections of this point. Now AB may be regarded as the trace, on the perspective plane, of the plane of the circle of shadow, and d' being the vertical projection of a point of this shadow, $d'd$, parallel to AB, represents a line of this plane of shadow, parallel to its vertical trace. The horizontal projection, pp'', of this line will therefore be parallel to the ground line, at a distance, $d'd$, back of it, equal to that of the point d' back of the perspective plane. This line (a little misplaced in plan, in this figure, to avoid that confounding of points which so often

Pl. X.



occurs in complex figures) pierces the horizontal plane, F′G′, at p', which is horizontally projected at p, and is a point of the trace of the plane of shadow on the plane F′G′. But OO′, where the vertical trace AB meets F′G′ is another point of this trace, hence Qp is the trace of the plane of shadow on the plane F′G′; and TT′, its intersection with the semi-circumference FKG—F′G′ is the point of shadow itself, whose perspective can at once be found, at t, by a diagonal and perpendicular (not shown).

A second method of finding t, is by purely perspective constructions, without recourse to its projections. Thus, through any point, as U, on a known circle parallel to the front one, draw Uy parallel to AB, and it will be the perspective of a line of the plane of shadow, parallel, really, and in perspective, to AB, and being in the plane of the small circle k'''UI, will meet the plane of the semicircle F′F″G′, in rk''', at y. Then O′y (not shown) is the perspective of the trace of the plane of shadow on the plane of F′F″G′, hence t, its intersection with the latter curve, is the perspective of the point of shadow on that semicircle.

Third; to find the point on the great semi-circumference, OK—LO′C. This point will be, as in the last topic, the intersection of the trace of the plane of the circle of shadow on the vertical plane OK, with the given semi-circumference. But OO′ is a point of the trace of the plane of shadow on any plane through OO′, and a line joining any point of AB, as mm', with any point of Op, as pp', will be a line of the plane of shadow, meeting the plane of OK—LO′C in a point, nn', of the desired trace. Revolving nn' to NN′, about a vertical axis at O, we have N′O′ as the revolved position of the trace in question, and P, the revolved position of the required point of shadow. A diagonal through the true position, on LC, of P, would obviously meet the perspective plane at P, and PD would be its perspective. PD meets O′E, the perspective of the perpendicular through the real position of P, at P′, which is the perspective of the required point on OK—LO′C.

Example.—*Find the lowest point of the shadow, that is, the point of contact of a tangent whose perspective shall be parallel to the ground line.*

The General Problem of the Sphere.

PROBLEM LXXXIX.

To find the perspective of the sphere, and of its curve of shade, when its centre is at the centre of the picture.

1°.—*To define the preliminaries, and find the perspective of the sphere.* Let S, Pl. XII., Fig. 83, be the centre, and S*s* the radius of the given sphere. Then the great circle, with this centre and radius, is the intersection of the sphere with the perspective plane, and is its own perspective. Let SD be the horizon; D, the vanishing point of diagonals; S, the vertical projection of the point of sight, and VR the vertical trace of a plane of rays, perpendicular to the perspective, or vertical plane, and which we will call the perpendicular plane. Also ME, perpendicular to VR, is the vertical trace of a plane of rays perpendicular to the former one, and also of the plane of the curve of shade. Now, revolving D to E, with the centre S, gives E as the point of sight, after revolution, about VR, into the perspective plane (51) and makes VR represent the perspective plane, made perpendicular to the paper. Then EP*s*′, tangent to the vertical projection S—*spt* of the sphere, is an element of the visual cone, in the plane of the paper; and S*s*′ is found as the radius of the perspective, *s*′T′T, of the sphere.

2°.—*To find the axes of the perspective of the curve of shade.* In order to render more manifest the independence of this construction of that of the other points of this curve, it is given before the construction of the latter points.

First: To find the *conjugate axis.*

Let GL, Fig. 84, be the ground line, and RL and R′L, a ray of light. Revolve this ray about its own vertical projection, R′L, and into the vertical plane, when R′R″ will be equal to GR, and R″L will be the revolved position of the ray. Let the light in Fig. 83, be parallel to R′L—RL, then SR and R′L′ will be parallel, and so will ER and R″L, and R, Fig. 83, will be the vanishing point of rays. Now, the circle with radius S*s*, is the revolved position of the one whose vertical projection is *s*S*t*. Drawing tangents—not shown—to this circle, and par-

allel to ER, we have A and B, the revolved positions of points of shade. a and b are their primitive positions. Then, observing that Aa is the perpendicular distance of a, in front of the perspective plane, that is, of the paper, A$'$ is the intersection of a diagonal, through a, with the perspective plane. A$'$D is the perspective of this diagonal, and aS, that of the perpendicular through a. Hence a' is the perspective of a, and is one extremity of the conjugate axis.

The perspective, b', of b,B, is similarly found. B may be found from A, by a diameter AB, without drawing a tangent at B.

Second: To find the *transverse axis.*

Bisect the conjugate axis, $a'b'$, at d, and XY, perpendicular to $a'b'$, will be the indefinite transverse axis. To find its extremities, conceive the point of sight to be projected upon the plane of shade, and tangents to be drawn from the point thus found, to the circle of shade. The projecting line of the point of sight will be in the perpendicular plane VR, which is perpendicular to the plane of shade; hence the chord of contact of the two tangents will be perpendicular to the trace VR; also, as those tangents were drawn from a projection of the point of sight they intercept the chord whose perspective will be the longest line in the perspective of the circle of shade; that is, its transverse axis. But the plane of shade is perpendicular to the direction of the light, and ER is the revolved position of the ray of light SR, passing through the eye. Hence So, perpendicular to ER, is the revolved position of the intersection of the plane VSR with the plane of shade, and is the distance of the projection, o, of the point of sight, E, upon the plane of shade, from the centre of the sphere. Hence, to avoid drawing another diameter, transfer So to So', and draw the tangent $o'p$, when p will be the extremity of a chord perpendicular to SE, and equal to the one whose perspective is desired. Finally, draw the visual ray Epq, which, as VR represents the perspective plane, when E represents the point of sight, gives Sq, equal to the semi-transverse axis required. Then lay off Sq each way from d, as at X and Y, and XY will be the transverse axis of the perspective of the curve of shade.

3°.—*To find any other points of the perspective of the curve of shade.*

First Method. By secant planes, and tangent rays.

The simplest conception of a point of shade upon a surface, is, that it is the point of contact of a ray of light with that surface. We shall now find the perspectives of points of shade thus constructed. Thus, FL is the vertical trace of a plane of rays perpendicular to the vertical plane, and the vertical projection of the small circle, cut from the sphere by that plane. Then the circle, partly shown, on FL as a diameter, is that small circle, revolved into the vertical plane, and the tangents to it at F′ and H, and parallel to the revolved ray of light ER, give F′, and H, as revolved points of shade. In counter-revolution, these points return to f and h, and these being their vertical projections, fS, and hS are the perspectives of the perpendiculars through them. Returning to the tangent rays, at F′ and H, these pierce the perspective plane at G and K, in the trace, FL, of the plane which contains them. Hence GR and KR, drawn to R, the vanishing point of rays, are their perspectives; which intersect the perspective perpendiculars, fS and hS, at f' and h', the perspectives of the points, f and h, of the curve of shade.

Any number of points could be found, as just explained. M and N, the extremities of the diameter, MN, of the curve of shade, in the perspective plane, and perpendicular to the projection SR of the light, are their own perspectives.

Second Method. By right angles inscribed in small semicircles whose diameters are traces of planes of rays, one side of the angle being the chord of the angle between the plane of shade and the vertical plane of projection.

Let the application be to any small circle, PQ, whose revolved position is PIQ. As already seen, AB, perpendicular to ER, is the revolved position of that diameter of the circle of shade, which lies in the plane VSR; hence BSR is the true size of the angle made by the plane of shade with the vertical plane, and Bt is its chord. Hence drawing the diameter UI, parallel to AB, in the circle on PQ, the angle QDI equals BSR, and QI is its chord. Also I is a revolved point of shade, as is U, likewise. EO, parallel to QI, is the revolved visual ray which meets the perspective plane at O, the vanishing point of all parallels to itself. Hence OQ is the perspective of IQ, and contains the perspective of I. Again, IP, the other side of the inscribed right angle PIQ, meets the perspective plane at P, and EV, parallel to it meets RSV, in V, the van-

ishing point of IP, and of all parallels to it. Hence VP is the perspective of IP, and meets OQ, both produced, at i, the perspective of I. Likewise VQ, the perspective of UQ (not shown) meets OP, the perspective of UP, at u, the perspective of U.

This method does not apply to the great circle st, since the vanishing points, O, and V, and the points to be found, a' and b', are all in the same line. The latter points are therefore found by diagonals and perpendiculars, as before explained.

The tangents from R to s'T′T, the perspective of the sphere, are the traces, on the perspective plane, of planes of rays, tangent to the circumscribing visual cone which is tangent to the sphere. Hence these planes are tangent to the sphere, in points, which are points of shade; and their traces are tangent to the base of the cone. But this base, being the intersection of the cone with the perspective plane, is the perspective of the sphere, hence those traces are tangent to the perspective of the sphere, at points, T and T′, of the perspective of the curve of shade. These points, moreover, are the limits of the visibility of the curve of shade.

Remark.—The perspectives of the sphere, and of a part of the curve of shade, are larger than their projections, because half of the sphere is in front of the perspective plane, and the apparent contour, and half of the curve of shade, are on the hemisphere, so situated.

PROBLEM XC.

To find the perspective of a sphere, whose centre is in the visual perpendicular; together with its curve of shade; and the shadow on the horizontal plane.

1°.—*To find the perspective of the sphere.*

Let E,S, Pl. XII., Fig. 86, be the point of sight, AB, the ground line, CD the horizon, and D, the vanishing point of diagonals. Let the centre of the sphere be behind the perspective plane, in the visual perpendicular, ES—S, at a distance equal to the radius of the sphere; and let the circle, with radius Ss, be the vertical projection of the sphere. Then make SS′ equal to this radius, and the circle with S′ as a centre, and radius

S′S, will be the vertical projection of the sphere, after revolution about the vertical axis ES*s*, the intersection of the perspective plane with a vertical plane, containing the point of sight and the centre of the sphere. In this revolution, the point of sight will proceed to D, the vanishing point of diagonals; and the tangents, of which D*t* is one, will be the revolved positions of the highest and lowest elements of the visual cone, which is tangent to the sphere in its circle of apparent contour, parallel to the perspective plane. Now, as ES*s* may be properly conceived as the trace of the perspective plane upon the plane of the paper, after the revolution just described, *t*′, the intersection of D*t* and ES*s*, determines S*t*′ as the radius of the perspective, *t*′*as*′*b*, of the sphere.

2°.—*To find the vanishing line of the plane of shade.*

The plane of the circle of shade of a sphere, is perpendicular to the direction of the light; hence its vanishing line (40) is the trace, on the perspective plane, of a visual plane perpendicular to the rays of light; hence it is only necessary to find one point of it. That point may be the intersection of the perspective plane with a visual ray perpendicular to a ray of light, since that visual ray is a line of the perpendicular visual plane, just described. Now, by revolving GSR, assumed as the meridian plane of rays perpendicular to the vertical plane, about its vertical trace, SR, and into the vertical plane, the point of sight, E,S, will appear at E′, on SE′, a perpendicular to GSR, at S, and equal to ES, or DS. But GSR being the direction of the vertical projection of the light, it will contain the vanishing point of rays, and the horizontal projection of the light not being given, that point can be assumed, as at R. Then RE′ is the revolved position of a ray of light through the eye, and E′G, of the visual ray perpendicular to it. Hence G, the intersection of E′G with the axis, GSR, is a point in the vanishing line of the plane of shade. Hence LGK, perpendicular to GSR, is this vanishing line.

3°.—*To find four points of shade; the extreme visible ones, and the other extremities of the diameters through them.*

Planes of rays, tangent to the visual cone of the sphere, will be tangent to the sphere at points of its apparent contour, and will contain a ray of light through the point of sight. Hence their traces on the perspective plane will pass through R, and will be tangent to the perspective of the sphere at the extreme

visible points of its shade. R*a*, and R*b* are these traces, and *a* and *b*, the extreme visible points of shade. Draw *a*S*d*, and *b*S*c*, which are the indefinite diameters through *a* and *b*. To find their limits, let ACDB, Fig. 85, represent the circle of shade, placed in the plane of the paper; let OA and OB represent the traces, R*a* and R*b*, and then ASD and BSC will represent the diameter just described; and OSI, the ray SR, and the diameter perpendicular to it, and in the plane GSR. Now observe that the chords, AC and BD, of the circle of shade, are parallel to the diameter just described. Hence the vanishing point of those chords is in the vanishing line, LGK, of the plane of the circle of shade, at G, the vanishing point of that diameter. Then, in Fig. 86, draw *a*G, and *b*G, the indefinite perspectives of the chords AC and BD, and they will limit the diameters *a*S*d* and *b*S*d*, at *a* and *d*, two additional points of shade.

4°.—*To find any four other points of shade.*

To do this, consider, first, that all lines in the plane of shade, have their vanishing points in the vanishing line, LK, of that plane; second, that if these lines are at right angles, and contain the extremities of any diameter of the circle of shade, they will intersect each other on the circumference of that circle.

To find the requisite vanishing points, revolve the visual plane which is parallel to the plane of shade, about its trace LK, into the perspective plane. E will then appear in the trace SR, at E″, by making GE″=SE or SD. Then any lines at right angles to each other, through E″, will meet LK in the necessary vanishing points. Let E″I and E″J be such lines, giving the vanishing points I and J, the latter beyond the border, on LGK produced. Then J*a* and I*d*, the perspectives of a pair of lines, perpendicular to each other through the extremities of the diameter *ad*, determine, by their intersection, the point of shade *e*. Likewise J*d* and I*a* determine the point *f*; which may also be found as the intersection of either J*d* or I*a* with the diameter *e*S*f*.

Proceeding similarly with the other diameter *bc*, the lines J*c* and I*b* (not shown) would give another point, and I*c* and J*b* (also not shown) a fourth point, as required.

5°.—*To find eight additional points.*

Returning to E″, draw another pair of lines as E″K and

E″L, at right angles to each other, giving the new vanishing points L and K. As the result of the previous operations we now have four known diameters, of which three are shown. Lines from L and K, drawn through both ends of each of these diameters, will give eight additional points, of which only a few are shown. Thus K*c* and L*b*, through the extremities of *bc*, give the point *g*; and K*b* and L*c* likewise give *h*; which, as in similar cases, is also the intersection of either K*b* or L*c* with the diameter *g*S*h*.

Other points are omitted, to avoid confusion; also, because eight finely constructed points, especially when, as at *a* and *b*, two of them are points of tangency, are quite sufficient to assist in accurately sketching an ordinary ellipse.

6°.—*To find the extremities of the chord, which is parallel to the perspective plane, and through* S.

This chord is the perspective of that diameter of the circle of shade, which is parallel to the perspective plane. Hence the point D, the line ES*s*, and the parallel diameter VV′, exhibit, in a side view, the true relative positions of the eye, the perspective plane, and the parallel diameter of the circle of shade. Hence, drawing the visual rays, of which DV is one, we have S*v* for the true size of the perspective of S′V. The line through S and perpendicular to SR, is the indefinite perspective of the diameter considered, and by laying off S*v* upon it, each way from S, as shown at *r*, we shall have the chord, *rr*′, required.

7°.—*To find the highest and lowest points of the perspective curve of shade.*

These points are the points of contact of tangents whose perspectives are parallel to the ground line. As none of the tangents, in space, to the curve of shade have this position, the required ones, must by (Theor. IX.) lie in planes which contain a visual ray parallel to the ground line. The intersection of this ray with the plane of shade will be common to the traces of these planes on the plane of shade.

See, now, Pl. XIII., Fig. 88, where A′B is the ground line, and CTO the horizontal projection of the sphere, since, as in Pl. XII., Fig. 86, the sphere is still supposed to be behind the perspective plane and tangent to it. C′P′ is the horizon, EE′ the point of sight, R, and H, the vanishing points of rays, and of their horizontal projections, respectively. The circle E′C′

is the perspective of the sphere, and *ijkl*, that of its curve of shade. SA—E′A′ is the diameter of shade, parallel to the perspective plane, and A, its intersection with the horizontal plane, is a point of the horizontal trace, AMN, of the plane of shade. Since this plane is perpendicular to the rays of light, its traces are perpendicular to the projections of those rays. Hence AMN is perpendicular to *h*E, the horizontal projection of such a ray, and the vertical trace, MK, is perpendicular to E′R, that is, parallel to E′A′. Next, we have EP—E′P′, parallel to the ground line, for the visual ray common to the visual planes tangent to the visual cone of the sphere, and containing the required horizontal perspective tangents. This visual ray pierces the plane of shade MMK, at P′,P, by an obvious construction.

Tangents, from PP′, to the curve of shade, will be the traces of the visual planes just mentioned, upon the plane of shade, and they will be the originals of the required perspective tangents. But we have not the projections of the curve of shade, and to avoid making them, we revolve the plane of shade NMK about its vertical trace MK, into the perspective plane. The circle of shade will then appear in its real form in the circle S′F, whose centre, S′, is at a distance, S′Y, from the axis, equal to the hypothenuse of a right triangle, whose base is E′Y, and whose altitude is ST. Likewise PP′ will fall at P″, at a distance, KP″, from the axis, equal to the hypothenuse of a right triangle, whose base is P′K and whose altitude is PB. Then tangents, as P″F, are the revolved positions of the originals of the required perspective tangents, and F, that of the point of contact of one of them with the circle of shade. In the counter revolution, Q, being in the axis, remains fixed, and P″ returns to P′; giving P′Q for the vertical projection of the tangent, on which F′, the vertical projection of F, is found by revolving F back, in an arc whose vertical projection is FF′, perpendicular to MK.

Now note that S′*g*, parallel to E′A′, is the revolved position of that diameter of the circle of shade, which is parallel to the vertical plane; hence F*g* is the distance of F′ behind the vertical plane CO, and F*g*+ST, laid off at F′*g*, therefore gives the intersection of the diagonal through F′ with the perspective plane. GD, the perspective of this diagonal, intersects F′E′ that of the perpendicular from F′, at *f*, which is the perspective

of F′, and is the point of contact of the horizontal tangent *ff′*, and hence is the highest point of the perspective curve of shade *ijkl*.

The lowest point can be found in the same way.

Remark.—The next two topics are auxiliary to the construction of the axes of the perspective curve of shade.

8°.—*To find, in Fig.* 85, *the originals of the perspective diameters ad and bc, of Fig.* 86.

Returning a moment to Fig. 85, we see that IJ is that diameter of the circle of shade, ACDB, whose perspective is the conjugate axis of the perspective curve of shade. Also, that the chords CJ and BI divide AD in a certain manner as at M.

It is now necessary to show how to locate the diameters, AD and BC, in their true position. To do this, conceive of the circle of shade as the base of a right cylinder of rays. Visual planes, tangent to this cylinder, will determine its extreme visible elements, and the feet of these elements are the last visible points, *a* and *b*, of the base. Also these planes will contain, in common, the ray of light through the point of sight; hence their traces on the plane of shade will contain the intersection of this ray with that plane.

Now S″, in Fig. 86, is the revolved position, on E′S produced, of the centre of the sphere, after revolution about the axis GSR; and E′R is the revolved position of the visual ray of light common to the tangent planes just described. Hence, as the plane of shade is perpendicular to the light, S″*o*, perpendicular to E′R, is the distance, in the plane of shade, from the centre of the sphere, to the intersection of the common visual ray with that plane.

Returning again to Fig. 85. This figure is here reduced, but, in practice, should be made in full size, that is with the circle ABDC, equal to a great circle, *s*S″S′, of the sphere, the circle of shade being a great circle. Then make SO=S″*o* from Fig. 86, and the tangents from O will determine the true relative positions of the diameters whose perspectives are *ad* and *bc*.

9°.—*To find the perspective of the horizontal trace of the plane of shade.*

One point of the required perspective trace, is the intersection of the perspective of any diameter of the circle of shade with the perspective of its own horizontal projection. The

latter line will contain the perspective of the horizontal projection of the centre of the sphere; which point must now be found. Make Ay equal to SS′, the true distance of the centre behind the perspective plane, and A will be the intersection of the diagonal through the horizontal projection of the centre, with the perspective plane. Then AD is the perspective of this diagonal, and it intersects, yS, the perspective of the perpendicular through the same point at s'', the perspective of the horizontal-projection of the centre of the sphere.

Let, now, the diameter chosen for finding the required trace, be the one parallel to the perspective plane. Then r'Sr, perpendicular to SR, is its perspective, as before shown, and s''P, parallel to the ground line, is its perspective horizontal projection. Hence, P, the intersection of these two lines, is a point of the required perspective trace.

To draw this trace, it is now only necessary to remember that, as it is a line of the plane of shade, its vanishing point is in the vanishing line, LK, of that plane; and, as it is a horizontal line, its vanishing point is in the horizon. Hence C is its vanishing point, and CP is the perspective of the horizontal trace of the plane of shade.

Remark.—Here it should be noted, for reference in a discussion presently to follow, that this line therefore contains the perspectives of the intersections, Q, etc., of all diameters of the circle of shade, with the horizontal plane. We shall first proceed with the construction of the *axes* of the ellipse *acfg*.

10°.—*To find the conjugate axis by means of the perspective division of lines, and knowing beforehand the pair of diameters through a and b.*

Produce the diameter *ad* to the perspective of its intersection with the horizontal plane, at Q, in the perspective trace CP. Then Qs'' is the indefinite perspective of the horizontal projection of *ad*, since all these diameters pass through the centre of the sphere, and s'', as shown, is the perspective of the horizontal projection of that point. Next, draw the verticals, dD′ and aA′, and A′D′ is the definite perspective of the horizontal projection of the diameter *ad*.

To find, now, the true size, and original, of this perspective horizontal projection, and the vanishing point of chords making equal angles with this original and the ground line, and therefore dividing both similarly. B, the intersection of D′A′

with the ground line, is one point of the original of D′A′, and S‴, the actual horizontal projection of the centre of the sphere is another point. S‴ is found by making yS‴=SS′ according to the given situation of the sphere. Then BS‴ is the indefinite original of D′A′. Now, with centre B, and any radius, describe an arc, as $u'u''$, and, through E, draw a visual ray parallel to the chord $u'u''$, which will give F, the vanishing point of the desired chords. Hence FD′ and FA′, produced to the ground line, give A″D″ for the true size of A′D′. Then transfer AM and DN, from Fig. 85, to A″M and D″N, and draw MF and NF, giving A′M′ and D′N′ for the perspective horizontal projections of these distances. Then the verticals, M′m and N′n, intersect ad at m and n, the perspectives of M and N in Fig. 85. Hence cm and bn, the perspectives of CMJ and BNI, Fig. 85, intersect GSR, the indefinite conjugate axis, in its extremities j and i.

11°.—*To find the conjugate axis*, ij, *by rectangular co-ordinates from vanishing points on* LK, *and having, already, the diameter* rr; *and, having the conjugate axis, to find the transverse axis.*

Find rr', as explained in (6°) then, as it is the perspective of that diameter of shade, which is parallel to the perspective plane, and perpendicular to SR, chords perpendicular to each other, through its extremities, and making equal angles with it, will meet at the extremities of the conjugate axis. The angles IE″G and JE″G, each being 45°, I and J are the vanishing points of such chords, and Ir and Jr' (not shown), would meet at j; and Ir' and Jr (also not shown) would meet at i, thus giving the conjugate axis.

To find the *transverse axis*, bisect ij, at k', and draw kl perpendicular to it, through k'. Take S″o, the distance, in the plane of shade, from the centre of the sphere to the ray of light through the eye, and lay it off from S′ to o'. Then, according to the explanations in (6°—7°) the tangent $o'p$ will give p, as the extremity of that chord of VSV′, whose perspective will equal the required transverse axis. For, VSV′, and o', thus represent—in the plane of the paper—respectively, the circle of shade, and the intersection of its plane with the visual ray of light perpendicular to it. But, in the same view, ES represents the trace of the perspective plane on the plane of the paper, and hence, drawing the transposed visual rays of

which Dqp is one, we find Sq equal to the semi-transverse axis. Then lay off Sq, from k', each way on kl, and kl will be the required transverse axis.

12°.—*To find both axes; independently of each other, and of any previously found points of shade; and directly from the vertical projections of their extremities.*

First: The transverse axis.

Conceive the circle of shade to be revolved about its own diameter, rSS$''$, till parallel to the perspective plane. Then sL$''$S$'$ will be its vertical projection, and o'', found by making So''=S$''o$, will be the true relative position of the point previously shown at o, and o', already described (8°; 11°). Hence tangents, as o''L$''$, will give the revolved positions of extreme visible points, as L$''$, of the circle of shade, *considered as an isolated circle.* The perspective of the chord joining these points, will therefore be the longest diameter, that is, the transverse axis of the perspective of the curve of shade.

Referring to the previous revolution, about the trace SR, when o was found, we see that o''', the primitive position of o, which is the same point as o'', is found by counter revolution in the arc whose vertical projection is $o''o'''$. The tangent, o''L$''$, meets the former axis, SS$''$, at a point beyond the circle, also indicated by S$''$. This point therefore is fixed, and o'''S$''$ is the vertical projection of the tangent o''L$''$S$''$. Also L$''$ revolves in an arc whose projection is L$''$L$'$, perpendicular to SS$''$, and which intersects o'''S$''$ at L$'$, the vertical projection of an extremity of the desired chord of the circle of shade, already described.

Returning now to the view where D is the point of sight, ES the perspective plane, and S$'$, the centre of the sphere; p is the same point as L$'$; hence the perpendicular distance from p to ES is the distance of L$'$ behind the perspective plane. Therefore make L$'$B$'$ a horizontal distance equal to this perpendicular distance, and B$'$ will be where the diagonal through L$'$ meets the perspective plane. Then the intersection of B$'$D, the perspective of this diagonal, with L$'$S, that of the perpendicular, through L$'$, will give l, one extremity of the transverse axis. The other extremity, k, can be similarly found; or, better, simply make $k'k$=$k'l$.

Second: To find the conjugate axis.

Revolve the great circle, contained in the plane GSR, about its own diameter, XX$'$, as an axis, and sL$''$S$'$ will be its vertical

projection. Tangents to $sL''S'$, parallel to $E'R$, will then evidently give the revolved positions, as I'', of the points of shade on this great circle, that is those whose perspectives are the extremities of the required conjugate axis, because the visual plane, E'—GSR, is perpendicular to the vertical plane, and through the centre of the sphere, and it hence divides the oblique visual cone, whose base is the circle of shade into symmetrical halves. In counter revolution, I'' returns to I', in the arc, whose vertical projection, $I''I'$, is perpendicular to GSR.

Recollecting that the diameter of the great circle XX' is the axis of the present revolution, a tangent to $sL''S'$ parallel to SR, and on the side towards E', will be the true relative position of the trace, on the paper, of the revolved perspective plane. Hence lay off the perpendicular from I'' to this tangent, on the horizontal line through I', at $I'D''$, to find the intersection of the diagonal, through I', with the perspective plane. Then i, an extremity of the conjugate axis, will be the intersection of $D''D$, the perspective of the diagonal, with $I'S$, that of the perpendicular. j can be similarly found.

13°.—*To find the axes; independently of any previously found points of shade: and by means of the visual cone, whose base is the circle of shade.*

The visual cone whose base is the circle of shade, is an oblique cone, having that circle for its circular base. The centre of the sphere, being in the visual perpendicular, the perpendicular plane, GSR, is perpendicular, both to the perspective plane, and to the plane of shade, and it divides this cone symmetrically. Hence it contains that diameter of the circle of shade, whose perspective is the required conjugate axis. That chord of the same circle whose perspective is the transverse axis, being perpendicular to this diameter, is parallel to the perspective plane.

See, now, Fig. 87, where like points, so far as this and Fig. 86 agree, have like letters. Fig. 87 is reduced only for want of room to make it of full size. As in Fig. 86, the circle SS''—the centre at S—is the vertical projection of the sphere, and the equal circle, with centre at S'', is the projection of the sphere after revolution about the trace SR as an axis. $E't$, then represents a tangent visual ray, and St', is the radius of the perspective of the sphere, since SR, with reference to E', is the trace of the perspective plane on the plane of the paper.

Also E′R is, as before, the revolved position of the ray of light through the eye. Then IJ, perpendicular to ER, is the revolved position of that diameter of the circle of shade which is cut from it by the plane SR. But E′I and E′J are the revolved visual rays from its extremities, and ij is its perspective, and the required conjugate axis. Bisect ij at k', and through k' draw kl, perpendicular to SR, for the indefinite transverse axis. The visual ray E′k' intersects IJ at K′, the original of k' the centre of the ellipse. Hence $k''l''$ perpendicular to IJ at k' is the chord whose perspective is the transverse axis. But this chord is, in space, parallel to the perspective plane; hence revolve it about K′ to the position K″L″, parallel to SR, and draw the visual rays, E′K″ and E′L″, which gives KL, equal to the perspective of the transverse axis. Hence, finally, lay off k'K or k'L each way from k' on kl, and kl will be the transverse axis required.

14°.—*To find the perspective of the shadow of the sphere on the horizontal plane.*

Points in this shadow will be the intersections of perspectives of rays of light, through the extremities of the perspectives of diameters, with the indefinite perspectives of the shadows of those diameters on the horizontal plane of projection.

The perspective of the shadow of the centre of the sphere on the horizontal plane, will be common to all those indefinite perspectives, and another point in each will be the perspective of the intersection of its corresponding diameter, produced, with the horizontal plane, in the trace TC.

Now s''H is the perspective of the horizontal projection, of the ray through the centre of the sphere, whose perspective is SR. Hence O, the intersection of these lines, is the perspective of the shadow of the centre of the sphere, and of the circle of shade, upon the horizontal plane. Also, the diameters ad and ef, for example, pierce the horizontal plane in points whose perspectives are Q and U, making QO and UO the perspectives of the shadows of ad and ef, on the horizontal plane. Then the perspective rays, aR, and dR, intersect QO at a' and d', the perspectives of the shadows of a and d; and, similarly, eR and fr (the latter not shown), intersect UO at e' and f', the shadows of e and f. Any number of other points can be similarly found, and the curve sketched.

15.°—*To find the perspective of the shadow of the sphere on the horizontal plane, by tangents.*

In the absence of any readily apparent easy construction of the axes of this shadow, the method by tangents is desirable, since a point of *known contact* is nearly as serviceable as two *ordinary* points, in sketching a curve.

First. The tangents at *a* and *b*, are also tangent to the shadow at *a′* and *b′* (Theor. XXI.).

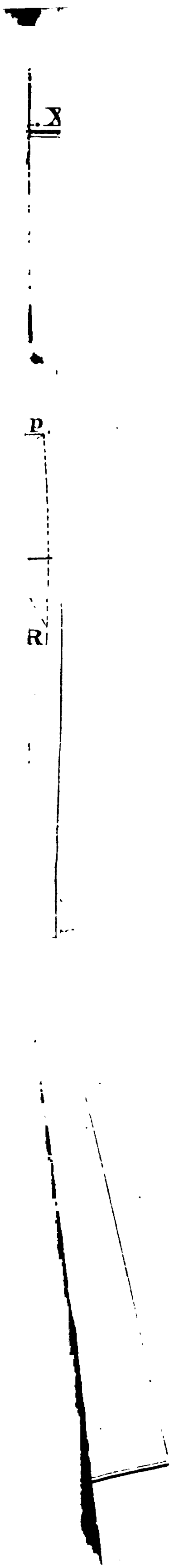
Second. *rr′*, having been exactly determined (6°) its shadow also can be exactly found, and as it is the perspective of the diameter perpendicular to *ij*, the tangents at its extremities are, in space, parallel to *ij*, and hence have G for their vanishing point. That is, G*r* and G*r′* are the perspectives of these tangents, and their shadows, which can readily be found, will evidently be tangents to the required curve of shadow, at the shadows of *r* and *r′*.

Third. The tangents at *i* and *j* are completely known, since they are parallel to *rr′*. Hence their shadows are, in reality, parallel to that of *rr′*, whose perspective is PO. But parallels have the same vanishing point, and these shadows, being in the horizontal plane, their vanishing point is the intersection of PO with the horizon, at a point, not shown, which we will call H′. Then lines from H′ to the intersections of the tangents at *i* and *j* with CP will be the perspectives of the shadows of those tangents, and will be tangent to the ellipse of shadow at their intersection with SR, the perspective of the rays from both *i* and *j*.

Fourth. A fourth pair of tangents is at *k* and *l*, which are known, being parallel to *ij*. Since *kl* and *rr′* are parallel in space, the perspectives of these shadows have the same vanishing point H′, mentioned in the last paragraph. Hence the indefinite perspective of the shadow of *kl* can be immediately found, and the shadows of *k* and *l* will fall on it at points found by drawing rays, *k*R and *l*R. The shadows of the tangents at *k* and *l* will join the shadows of *k* and *l* with the intersections of those tangents by CP.

Having thus the points of contact of eight fully determined tangents to the perspective ellipse of shadow, *a′b′c′*, it can be very accurately sketched.

16°.—*To find two diameters, and thence the centre of the perspective of the shadow of the sphere upon the horizontal plane.*


X
p
R

First. By Theorem XI., lines which meet in a visual plane parallel to the perspective plane have parallel perspectives. And the intersection of a line with the horizontal plane, is a point of its shadow on that plane. Hence we wish to find a diameter of the circle of shade, which will pierce the horizontal plane in the horizontal trace of a visual plane parallel to the perspective plane. To do this, lay off ES, the distance of the eye from the perspective plane, from y downwards, as shown in a reduced scale in Fig. 86*a*, which will give E‴, the projection of the eye on the horizontal plane of projection (E being on the horizontal plane through the horizon).

Again; as the horizontal trace of the plane of shade meets the perspective plane in the ground line, its perspective CP does so also, as at u, which is thus one point of the horizontal trace itself. Now rr', is the vertical projection, as well as perspective of that diameter of shade which is parallel to the perspective plane. And, making yS‴=SS′, the line S‴P′, parallel to the ground line, is its horizontal projection. Hence it pierces the horizontal plane at P′, which is therefore another point of the horizontal trace of the plane of shade. P′uY (see also Fig. 86*a*) is this horizontal trace and it meets the trace E‴Y, parallel to AB, of the parallel visual plane, at the point indicated by Y in both figures. Now, by the Theorem (XI.) the perspective of the diameter, through Y, will be parallel to CP, and as the perspectives and vertical projections of all the diameters of shade coincide, by (Theor. X.) Y′S will be parallel to CP and will be the perspective of this diameter.

The perspective of the shadow of this diameter passes through O and its intersection with CP, which is at an infinite distance. Hence a line through O, and parallel to SY′, will be that perspective shadow. Now the shadows of tangents to the circle of shade from Y begin at Y, hence the perspectives of those shadows will be also parallel to the shadow of SY′, that is, as just shown, to SY′ itself. The chord of the points of contact of these parallel tangents will be a diameter of the ellipse $a'b'c'$. And these points of contact will be the perspectives of the points of contact of the perspective tangents, themselves, with $abdc$, which can be constructed, since that ellipse is fully given by its axes.

Second. Among all the diameters of the circle of shade, one will have a shadow parallel to the ground line, as indicated by

the shadow OW, parallel to AB; since the perspective of such a shadow will be parallel to the shadow itself, and the shadows of all the diameters begin in the perspective trace, CP, of the plane of shade.

Parallel lines have parallel shadows, hence the perspectives of the shadows of tangents to the circle of shade, parallel to the supposed diameter, will be parallel to WO, the perspective shadow of that diameter. Now WS is the perspective of the diameter, and the ellipse *abdc* being fully given by its axes, the tangents to it, parallel to WS, can be exactly constructed by the methods given in plane problems on the ellipse. Their perspective shadows will begin at their intersections with CP, will be parallel to WO, and will be tangent to the perspective ellipse of shadow, at the perspectives of their points of contact with the perspective ellipse of shade *abdc*. The chord of these points of contact will be a second diameter, whose intersection with the former will be the centre of the ellipse of shadow.

Remarks.—*a.* In stating that there seems to be no obvious construction of the *axes* of the ellipse of shadow, it is meant that none appears obvious, by the principles of projections, or of perspective. They can, however, be constructed by plane geometry. For, having one diameter, a line bisecting it and one or more parallel chords, will be its conjugate, and plane geometry affords constructions for the axes, when a pair of conjugate diameters are known. We have thus the means of constructing this ellipse with any required degree of practical accuracy, in case, for example, it were drawn on a large scale.

b. All the diameters, as *ef*, between CS and Y′S, Pl. XII., Fig. 86, pierce the horizontal plane behind the parallel visual plane E‴Y; (Fig. 86*a*) hence the perspectives of those intersections are (Theor. XIV.) beyond C.

c. Without making the construction, it may be understood that the perspective of a sphere, whose centre is in any position, and of its curve of shade, and its shadow on any surface, can all be very easily found from their projections, by the method of three planes.

d. The perspectives of the sphere already constructed have been circles, because they were sections of the visual cone, parallel to its base, which was a circle. The visual cone of a sphere is, indeed, always one of revolution. If the centre of the sphere be out of the perpendicular from the eye to the per-

spective plane, while the perspective plane still cuts all the elements of the visual cone, the perspective figure will be an ellipse. But if the perspective plane is parallel to a tangent plane to the cone, the perspective figure will be a parabola, or, if it cut both nappes of the same cone, the perspective will be a hyperbola.

In all these cases, the perspective will *appear* circular, as it should; since the sphere itself does, when the eye is actually placed at the point of sight. The parabolic and hyperbolic perspectives correspond to cases, where the visual angle is 180° or more; that is where only a part of the sphere is really visible from the point of sight.

PROBLEM XCI.

To find the perspective of a skeleton groined arch, and of its shadows.

1°.—*The description of the arch.* The groined arch, here taken, Pl. XIII., Figs. 89–90, is formed of two semi-cylinders, whose axes are in the same plane, and perpendicular to each other, and which are of equal diameter. Then in the reduced plan, Fig. 90, added to aid in conceiving of the positions of the points and lines, shown perspectively in Fig. 89, AB is the horizontal diameter of one of these cylinders, and BD is that of the other, and B and C are points of their intersection. Now if $Bh = Bf'$, the heights of h and f', in space, above the plane of the axes, that is, of the paper, are equal; and hh' and $f'h'$ are elements, one on each cylinder; in the same plane parallel to the paper, and equal to each other, as are BD and CD. Hence, h' is a point of the intersection of the cylinders. Also,

$$Bf' : f'h' :: BD : DC.$$

That is $Bh'C$ is a straight line, and is the horizontal projection of the intersection of the two semi-cylinders, since any other point than h' could have been similarly located. Hence that intersection is a plane curve, viz., a semi-ellipse.

But when cylinders thus intersect, they do so in two equal ellipses; hence, in the present case, AD is a second semi-ellipse of intersection.

This being settled, call the cylinder whose elements are par-

allel to Xx, the parallel cylinder, it being parallel to the ground line, and call the other the perpendicular cylinder. Then all the elements, as $e'f'$, of the parallel cylinder, which are cut off by both the vertical planes, AD and BC, are omitted, as at $g'h''$, between those planes; and all the elements, as hn, of the perpendicular cylinder, are likewise omitted, as at $h'r$.

2°.—*The perspective of the general outlines of the arch.*

Let the front face, AB, be in the perspective plane, as at ABA'B', A''G'B'', Fig. 89. It will then be its own perspective. Let AB be the ground line; EF, the horizon; V, the vanishing point of diagonals; R, that of rays, and H, that of their horizontal projections; and let E be the centre of the picture. Then AB, being a side of the square whose corners are the feet of the skeleton columns, AA'', etc., the perspective, ABCD, of this square, and of its inscribed circle, are found as explained before (Prob. LXIV.).

Fixing AA'' as the height of these columns, we describe the semicircle A''G'B'', the front line of the arch, on A''B'' as a diameter.

We can now, by diagonals and perpendiculars, rapidly find groups of twelve points each, by considering that horizontal planes, above the springing plane A''B''C'D', will cut each of the *four* arch faces, and the *two* ellipses, in two points, which are readily connected by parallels to the perspective plane, by perpendiculars, or by diagonals.

Thus, let ef be the trace of such a plane on the perspective plane, cutting the front vertical lines at e and f, and the front semicircle at g and h. Then the diagonal, eV, and perpendiculars, gE and hE, determine g' and r, the perspectives of the same points in Fig. 90. Then the parallel $e'f'$, through g', intersects the perpendiculars, eE, hE, and fE, at e', h', and f'; which are the perspectives of the same points on Fig. 90. Likewise, the parallel $r'y'$, through r, intersects the perpendiculars eE, gE, and fE, at y', y, and r', which are also, as before, the perspectives of the same points on the plan.

We have thus ten points already, omitting those, not shown, on the semicircle C'D', whose perspective is the semicircle, whose diameter, C'D', joins the intersection of the back columns, CC' and DD', with the perpendiculars, A''E and B''E.

3°.—*To construct various particular points in the outlines of the arch.*

First: to construct the perspective of the summit, X.

This point is the intersection of the perpendicular G′X and diagonal AD, Fig. 90. Hence, noting G′, Fig. 89, as the middle point of A′B′, X is the intersection of G′E, the perspective perpendicular, with the perspective diagonal A′V. And, as the original AD, Fig. 90, of this diagonal, is tangent to the elliptic edge of the groin, their perspectives will be tangent at X.

Second: to find points of the side curves, in the parallel through X.

This parallel X*x*, Fig. 90, is in the same parallel vertical plane with the points *p* and *q*, Fig. 89, where the circle, *k*G*qm*, is tangent to the ground square, ABCD, of the arch. Hence *x*, Fig. 89, the perspective of *x*, Fig. 90, is the intersection of the vertical *qx*, with the parallel X*x*. A similar point may be found on the left side curve.

Third: to find points vertically over some points of the ground circle *k*G*m*.

Since this circle is determined in part by points, as *k*, on the diagonals of the ground square; and since the ellipses, A″XD′ and B″XC′, Fig. 89, are in vertical planes on these diagonals; points of these ellipses will be vertically over the points *k*, *o*, *n*, and *m*, of the ground circle.

To find these points, note K′, where the perpendicular KK′ meets the ground line—AGK being = 45°, so that GK shall be the diagonal AD, in place. Project K′ at K″, and draw the horizontal line K″K‴ in the front face of the arch. Then the intersection of the perspective diagonal, K‴V, with the perpendicular K″E (not shown), or with the vertical, *kk*′, gives *k*′, the perspective of the point over *k*. The points over *o*, *m*, and *n*, can be found in the same way.

Fourth: to find the highest points of the elliptic and side curves.

Make E′F=EV, and F, the line BB′, and the semicircle A″G′B″ will show, in a side view, the true relative position of the eye, the perspective plane, and the parallel cylinder. Then a tangent, from F to A″G′B″ will be the trace on the paper (perpendicular, for the moment, to the perspective plane BB′) of a visual plane, tangent to the parallel cylinder. Now the element of contact of this plane, projected at S, is the highest visible element of the parallel cylinder, and therefore contains

the highest points of the perspectives of all curves as AD or AC, Fig. 90, lying on that cylinder. S′, where SF meets the side view, BB′, of the perspective plane, is the perspective of this highest visible element; which, on returning to the paper as the perspective plane, gives S′*u*, parallel to the ground line, for the proper perspective of this element. In like manner, after counter-revolution, the horizontal line, through S, is the true vertical projection of the same element. Hence the perpendiculars, *a*E, UE, SE, and *c*E, intersect S′*b* at *b*, *u*, *s*, and *d*, the highest points of the perspectives of the diagonal and side curves, and the perspectives of the really highest points, visible from E (F) and whose vertical projections are *a*, U, E, and *c*.

Another view of this construction is too interesting to be passed by. The tangent visual plane to the parallel cylinder, contains all the tangent lines, to curves lying on that cylinder, at the points where those curves cross the element of contact of the tangent plane. But the trace, S′*b*′, of this plane, is the perspective of all lines contained in the plane, and as the plane is tangent to the parallel cylinder, that trace is parallel to the ground line. Hence by (Theor. IX.) S′*b* is the perspective of the four tangents, to the side and diagonal curves, at their highest visible points. Hence S′*b* is tangent to the perspectives (Theor. VI.) of those curves.

We pass now to the shadows of the arch.

4°.—*To find the shadows on the plane of the rear lines*, CC′, *and* DD′.

Having R, the vanishing point of rays, and H, that of their horizontal projections, BH is the perspective of the horizontal trace of a vertical plane of rays through the edge BB′, that is, it is the perspective of the shadow of BB′ upon the horizontal plane. This shadow meets the floor line, D*b*‴, at *b*‴, whence the vertical line, *b*‴*b*′, limited by the perspective, B″R, of the ray through B″, is the shadow of BB″ on the plane D′D*b*‴. At *b*′, begins the perspective of the shadow of the front semicircle A″G′B″. This shadow is, in reality, equal to that front semicircle itself, being found on a plane parallel to that of A″G′B″. Moreover, since the plane of the shadow is parallel to the perspective plane, the perspective of this shadow will be circular; having its centre at the perspective of the shadow of G″. Now *c*″, the centre of the perspective back semicircle, is the perspective of the projection of G″ upon the rear plane

CDD′. Hence $c''d'$, parallel to ER, is the perspective of the vertical projection (Theor. X.) of the ray through G″, and G″R is the perspective of the ray itself. Hence d', the intersection of these lines, is the perspective of the shadow of G″; and $d'b'$, which must be parallel to the ground line, is the shadow of G″B″. Then the arc b'Q, with centre d', is the shadow of the front semicircle A″G′B″, on the vertical rear wall D′Db''', and is limited by the horizontal line D′Q, which is the upper limit of that wall.

5°.—*To find the shadow of the front semicircle on the surface of the parallel cylinder, produced.*

Since the shadow of a circle, on a plane parallel to it, is an equal circle, the simplest solution for this topic, is by the familiar method of auxiliary shadows. Thus, any vertical plane, parallel to CDD′, and a little in front of it, will cut an element parallel to the ground line, from the parallel cylinder, and will contain a circular shadow of A″G′B″. The intersection of this shadow with that element, will be a point of the required shadow. It now remains to make the perspective of the parts just indicated. To avoid the ill defined intersections, caused by assuming the trace of the parallel plane, close to Db''', and erecting a perpendicular from its intersection, Z (not shown), with BD, to meet the almost straight curve from D′ upwards, we proceed as follows: Take At, for the real distance of the auxiliary parallel plane, in front of the rear wall CDD′, and t will be where a diagonal, through the supposed point Z, will meet the perspective plane. Then erect the perpendicular tt', and t' will be in a diagonal with the point where the parallel plane meets the side semicircle, B″dD′. Hence the diagonal, t'V (not shown), will meet D′dB″ in a point, t''', of the element cut from the parallel cylinder by the parallel plane. Or, project t' to t'', and the perpendicular, t''E, will determine the same point.

Next, to find the circular shadow. By laying off G″I=Bt, and on the horizontal line G″A″ produced, we shall find I, beyond the limits of the diagram, where a diagonal, through the centre of the semicircle, equal and parallel to A″G′B″ and in the parallel plane, cuts the perspective plane. Then the perspective, IV, of this diagonal, intersects the perspective perpendicular, G″E, in I′, the perspective of the centre of the parallel semicircle just described. Hence, I′d'', parallel to ER,

the perspective of the projection of the ray through G″ upon the parallel plane, meets G″R, the perspective of the ray itself, in *d*″, the perspective of the shadow of G″ upon the parallel plane. And *d*″*b*″, parallel to AB, and limited by the ray B″R, is the perspective of the shadow of G″B″ on the parallel plane. Hence the arc with centre *d*″, and radius *d*″*b*″, is the perspective of the required auxiliary shadow, and *q*′ its intersection with the element, *t*‴*q*′, before found, is one point of the required shadow of A″G′B″ upon the parallel plane. Other points can be similarly found.

6°.—*To find the shadows on the floor, and side wall*, D′DD″.

AH is the perspective of the indefinite shadow of AA″ on the floor. At D″, its intersection with the floor line DD″, draw the vertical line D″*a*′ limited at *a*′ by the ray A″R, and D″*a*′ is the shadow of a part of AA″ upon the side wall. Above *a*′, a curved shadow is cast by the front semicircle A″G′B″, points of which are found as follows: Assume any point, N, on the front semicircle, and let fall the perpendicular, NN″, upon the ground line. Then N″H is the indefinite perspective shadow of NN″ upon the floor, meeting the floor line DD″, at N‴. The vertical line N‴N′ is the shadow of NN″ upon the side wall, and it is limited at N′ by the perspective ray NR; giving N′ as the shadow of the assumed point N. Other points, being found in the same manner, and connected, will give the required shadow on the side wall.

7°.—*To find the shadow of the front semicircle*, A″GB, *upon the interior of the perpendicular cylinder.*

The tangent at T, parallel to ER, is the vertical trace of a plane of rays tangent to the perpendicular cylinder, and therefore marks the point T, where the required shadow begins. To find other points, assume any secant W*h*, also parallel to ER, for the vertical trace of a plane of rays, cutting the cylinder in two elements, whose perspectives are WE, not shown, and *h*E. The foremost point, W, of the former element, casts a shadow on the latter at *v*, the intersection of the ray WR, with the element *h*E. Other points being found in like manner, and joined, will give the required shadow. Only that part of the shadow, in the figure, falling between the curves A″G′B″ and B″XC′ is real.

8°.—*To find the shadow of the left side semicircle*, A″*b*C′, *upon the interior of the parallel cylinder.*

In constructing this shadow, use is made of the vanishing

point of the projections of rays upon planes parallel to the side circles of the arch. The vertical line, ER′, is the vanishing line of all such planes, and therefore contains the required point. But the trace, on the perspective plane, of the visual plane of rays, perpendicular to the planes of the side circles, also contains the vanishing point sought. This trace is RR′; hence R′ is the vanishing point of the projections of rays on all planes perpendicular to the ground line.

A tangent, R′*m*′ (a little displaced to avoid confusion), from R′ to the side circle, is the perspective of the trace, on the plane of that circle, of a plane of rays, tangent to the parallel cylinder. Hence *m*′, its point of contact, is the beginning of the shadow of that circle upon the parallel cylinder. The point *m*′ is, however, very vaguely determined by this construction. We therefore propose another. ER′ represents the projection of the ray of light, ER upon the perpendicular plane R′EL′, and EV equals the true distance of E in front of the perspective plane. Therefore VR′ is the true direction of the projection ER′ after the revolution of the perpendicular plane, into the paper, around LL′ as axis; and, recollecting the square plan of the arch, and the equality of its semicircles, A″G′B″ represents, at the same time, the relative position of a section of the parallel cylinder. Then draw the tangent at M, parallel to VR′, and M will show the real height of the element of contact of a plane of rays with the parallel cylinder. M″MM′ is then the vertical projection of this element, MM′ is its true distance behind the perspective plane, and *a*‴ is the vertical projection of its extremity, in the left hand side circle. Then make *a*‴M″=MM′ and M″V will be the perspective of the diagonal from the point whose vertical projection is *a*‴, and *m*′, its intersection (well defined) with A″*b*C′ is the perspective of *a*‴, the first point of the required shadow.

To find other points of the same shadow, draw any secant, as R′O, cutting from the side circle a point as O; whose shadow falls on the element as O′P of the parallel cylinder, contained in the secant plane of rays parallel to its axis, and having R′O for its perspective trace on the plane of the side circle. Then the ray OR meets this element at P, the shadow of O. Other points being likewise found and connected, we have *m*′P for the required shadow, of which only the part inked as a full line is real.

EXAMPLE.—*Reconstruct* O *and* O′ *as* m′ *was; beginning with a secant parallel to the tangent at* M.

91. The surface of the piedouche, a fragment of which is represented in Pl. II., Fig. 12, presents an example of a surface having the remarkable property of opposite curvatures; by which is meant, that all planes, *perpendicular to its axis*, intersect it in curves which are *concave* towards the axis, while all planes *through its axis*, intersect it in curves which are *convex* towards that axis.

92. From this fundamental property, it results, that some of the elements of the visual tangent cone, will be tangent to the exterior side of this surface, and others to its interior side. Only the former contacts can be real for a solid piedouche; hence, in such a case, a part only of the total apparent contour, geometrically considered, is real.

93. In order to examine these curious results with more advantage, let the surface of opposite curvatures be completely closed. The annular torus presents such a surface. This surface is generated by the revolution of any closed curve, usually a circle, about an axis exterior to it; but in its own plane.

Thus, see Pl. XIV., Fig. 97, where the extremities, A,A′, and A,B′ of the vertical diameter of the generating circle describe horizontal circles which divide the entire surface into two nappes. Of these, the exterior nappe, generated by A′D′B′ is wholly convex, or, *doubly convex*, while the interior one, generated by A′C′D′, is *concavo-convex*, being *concave* towards the axis in all its *horizontal* sections, and *convex* towards the axis in all its meridian sections.

Before proceeding with the problem of the perspective of the torus, some outlines, merely, of principles, and examples of the simpler theorems, belonging to the less familiar subject of the Theory of Curvature of Surfaces, will here be rehearsed.*

94. Through any three points, not on the same straight line, a circle can always be passed.

* Articles (94–101) are inserted to make this work the more conveniently complete in itself. Until the appearance of the Author's new Descriptive Geometry, the student must be referred to the French of *Leroy*, *Olivier*, or *Gournerie* on Des. Geom. or to any of the fuller analytical works, for a complete treatment of the theory of the singular points, $o''o'$; v,v', etc., at which the visual rays are tangent not only to the surface of the torus, but to the curve of apparent contour itself.

If there be *three consecutive points*, on any curve whatever, whether of single or of double curvature, the circle passed through them is called the *osculatory circle* of that curve, and its centre and radius are called the *centre, and radius of curvature* of that curve, for the three given points.

If any two curves, whatever, have thus three consecutive points in common, they will have the same osculatory circle, and are therefore said to be osculatory to each other.

95. The radius of a circle is a normal to its circumference, and, hence, to any curve with which that circumference coincides, at the place of coincidence. Hence a centre of curvature of any curve, may, for convenience, be defined as the intersection of two consecutive normals to the curve.

It follows, from the preceding, that each point of any curve, will, in general, have a new osculatory circle, and centre of curvature.

96. Passing to *surfaces;* any two surfaces are said to be osculatory, when every plane, containing a common normal to them, at the same point, cuts them in two curves which are osculatory to each other, at that point, or, in other words, have the same radius of curvature at the same point.

97. Now, for *surfaces in general, no two* of these normal planes will cut the given surfaces in curves whose radii of curvature will be the same; while any two normal planes, through a given point on a *sphere*, do cut equal circles from it, having, therefore, the same radius of curvature, viz., the radius of the sphere. Hence a sphere cannot be osculatory to any other surface.

98. Every curved surface possesses this remarkable property. *Among all the normal planes, passed through the normal, N, to a surface, at any point, P, upon it, there will always be two, perpendicular to each other, whose intersections with the given surface are called principal sections. And the radius of curvture of one of them, at P, will be a minimum, and of the other, a maximum.*

99. Thus, taking a common cylinder for illustration, and drawing a normal to its surface, at any point; one plane, through that normal, will cut the surface in straight elements, whose radius of curvature is therefore infinite; and another plane, perpendicular to the former, and hence to the axis, will

contain a right section, whose curvature, at the foot of the normal, is less than that of any other section made by a plane containing the normal.

100. Again, see Pl. XIV., Fig. 96, showing an elliptical warped hyperboloid, whose gorge is the ellipse ABCF—A′B′, and whose generatrix is GE—G′E′. Now let the radii of curvature, which are estimated *towards* the centre, O,O′, as are those of the gorge and of all the elliptical sections, be called *positive*. And let those of the meridian curve, and other hyperbolic sections, be called *negative;* since they are directed *from* the surface, outwards, or *from* the centre O,O′. Then, for all sections, made by normal planes at C,O′, for example, planes which therefore contain the normal CO—O′, the radii of curvature vary from that of the gorge at the vertex C,O′, which is *positive* to that of the hyperbolic meridian, in the vertical plane through CF, which is negative. Now all the planes, from CD′Z′ to the one containing the generatrix GE—G′E′, will cut the given surface in *hyperbolas*, which will evidently be more and more *obtuse* at C, and all planes from the latter one, to that of the gorge, will contain *elliptical sections*, which will be more and more *acute* at C.

The curvature of the generatrix, is $= +\infty$ in the sense CO and $= -\infty$ in the sense CH. Now put $OA = a$, $OC = c$, for the axes of the ellipse of the gorge, AFC; and $OC = c$, and $O'b = b =$ the definite conjugate axis, limited when made vertical at C,O′, by the asymptotes to the hyperbola lying in the plane CD′Z′. Then, finally, we have from analysis, the radius of curvature of the ellipse, at C, will be a line $CI = R = \frac{a^2}{c}$; and that of the hyperbola at the same vertex is a line $CH = -R' = -\frac{b^2}{c}$.

We have thus the radii

$$+\infty; \quad +R; \quad -R'; \quad -\infty$$

where R is the *least* of all the positive radii and $-R'$, though numerically least, is, taking account of the signs, algebraically, the *greatest* of all the negative radii. Hence the theorem is proved for the general, or elliptical warped hyperboloid, and may serve to indicate the very similar, but a little simpler, proof for the general ellipsoid. Then, recollecting that all surfaces are doubly convex like ellipsoids, or concavo-convex, like warped

hyperboloids, we may infer, for now, the truth of the theorem in all cases.

101. The two following principles of the theory of curvature are also employed:

First. For each point, P, *of any surface, there is an osculatory surface* (94) *of the second order*, which, for a *concavo-convex surface*, like the interior half of a torus, will be a *warped hyperboloid.* (Des. Geom. .)

Second. The *element* of a circumscribing tangent cone to any surface, S, and the *tangent to its curve of contact*, at the point of contact, T, of the element, determine a *tangent plane* to the surface at T; and *these lines are parallel to the conjugate diamters of the section of the osculatory surface*, at T, *made by a plane parallel to the tangent plane.* (Des. Geom. .)

Problem XCII.

To construct the perspective of an annular torus.

Let CD—C′A′D′B′, Pl. XIV., Fig. 97, be the generatrix of the torus, and O—O′O″ its axis, which is taken vertical; and let E,E′ (at the right of the plate) be the point of sight.

We will first, in a series of topics of convenient length, find the ordinary, and the peculiar, points of the apparent contour.

1°.—*To find the points of contour on the visual meridian plane.*

As this plane is here taken parallel to the vertical plane of projection, it is only necessary to draw the four tangent visual rays, whose vertical projections are E′g', E′h', E′b' and E′a', and whose horizontal projections all coincide with OE, the horizontal trace of the given meridian plane. Then, by projecting down g', h', b', and a', at g, h, b, and a, we have the four points of contour required.

If the plane EO had not been parallel to the vertical plane, it would have been revolved to that position, or into a horizontal plane, and then, having found the revolved positions of the required points, a counter-revolution about the axis of the torus would have given their true position, as has been explained in Probs. XIII. and XIV.

The construction of the points on the meridian, parallel to

the vertical plane, would then have been a distinct topic, solved as the present one has been, only that the tangents from E′ would then have properly been called vertical traces of visual planes, perpendicular to the vertical plane of projection.

2°.—*To find points of contour on the greatest and least horizontal circles of the torus.*

These points are first shown in horizontal projection, as the points of contact of tangents from E, with the circles whose radii are OD and OC. By a familiar plane problem, these points of contact are *d*, *u*, *u″*, and *d″*; which are the intersections of the circles, just named, with the circle having OE for its diameter, and J for its centre.

The vertical projections of the greatest and least circles, are F′D′ and H′C′, on which *d″* and *d* are projected, at *d′* and *u′*. If the plane OE were not parallel to the vertical plane of projection, *d* and *d″*, also *u* and *u″*, would have separate vertical projections. The points, themselves, *d″ d′*, etc., are defined as the points of contact of vertical tangent visual planes, with the torus.

From the situation of the first four points, *a*, *a′*; *b*, *b′*, etc., it is evident that there are no points of apparent contour on the highest and lowest horizontal circles, whose radii are equal to OA.

3°.—*To find other points of each branch of the contour.*

These points are found by means of auxiliary tangent cones, having the same axis as the torus, and tangent to it in horizontal circles. To abbreviate the construction to the utmost, let a series of four such cones have a common base in the horizontal plane E′R′, through the point of sight, and let OR be the radius of the horizontal projection of this base. Then, from R′, the vertical projection of R, four tangents to the meridian curve contained in the meridian plane OE, will be the elements, parallel to the vertical plane, of these cones, and OR will be their common horizontal projection. R′F‴, R′K′, R′N′, and R′C‴ are these four tangents, and their intersections, not noted, with O′O″, would be the vertical projections of the vertices of the four tangent cones. If, now, a visual plane be drawn, tangent to each of these cones, the intersection of its element of contact, with the circle of contact of the cone and torus, will be its point of contact with the torus; that is, a point of apparent contour. Two such planes can be drawn to

each of the four cones, giving eight points in all. The common trace of these planes, on the plane O″E′, of the common base of the cones, will be tangent to that base, and will contain EE′. But tangents from E, to all circles having O for their centre, will find their points of contact in the circle on OE as a diameter; hence W is the point of contact of such a tangent, to the base, O—RW, of the cones, and WO is the horizontal projection of one element of contact, on each of the four cones. Another tangent, symmetrical with EW, relative to the meridian plane EO, will be the common trace, on the plane E′O″, of the other four tangent planes, whose elements of contact are horizontally projected in fOc'', symmetrical with WO. Now F‴t; K′q; N′s, and C‴r, are the radii of the circles of contact of the four cones with the torus, then the horizontal projections f''F″f, etc., of these circles, intersect the elements before found, in the points f'',f' and f,f'; k'',k' and k,k'; n'',n' and n,n'; c'',c' and c,c'.

4°.—*To find the points of contour on the meridian plane, perpendicular to the vertical plane.*

These points are found on the principle (Shades and Shadows, 80) that the traces, on any meridian plane, of visual planes perpendicular to it, and tangent to the torus, will be tangents to the meridian curve in that plane, at the points of contact of the visual planes with the torus, and that these traces will contain the projections of the point of sight on the meridian plane.

To realize the preceding principle, it is sufficient to conceive the system of the figure to be revolved 90° about the vertical axis of the torus. Then O″ becomes, evidently, the point of sight; and O″E‴, and O″M′, are the vertical traces of two visual planes, perpendicular to the vertical plane, and tangent to the torus. Hence E‴, and M′, projected horizontally at E″ and M, are the points of contact of these planes. Then, counter-revolving the system to its given position, M,M′, and E″E‴, return to m,m' and e,e'; and the opposite similar points on the right of O′O″, to m'',m' and e'',e'.

5°.—*To find the singular points of the interior contour.*

These are the points where the visual rays are tangent, not only to the inner half of the torus, but also to the curve of contour itself. They can be approximately detected by inspection, as in the analogous problem in curves of shade (Shades and Shadows, Prob. XXVIII.) at o'',o' and o,o'; v'',v' and v,v'.

But as they are the limits between the visible and invisible, that is, real and imaginary, portions of the geometrical apparent contour, it will be interesting to find them by exact construction.*

To avoid confusing the figure, see the separate Fig. 98, Pl. XIV., in which, however, the same point of sight, E,E′, is retained for convenience. P—Lp' is, here, the vertical axis of the torus, and the semicircle OC—O″C′o' is the generatrix of its interior half. Having given, then, the point of sight E,E′; to find the point where the visual ray is tangent, at once, to the torus, and to its apparent contour.

To find this point, we proceed with the problem in an inverse order, by means of the following operations:

First. For each, of several assumed points, on a meridian of the torus, construct a visual ray, that would be tangent, at once, to the torus, and to the apparent contour as seen from a point in that ray.

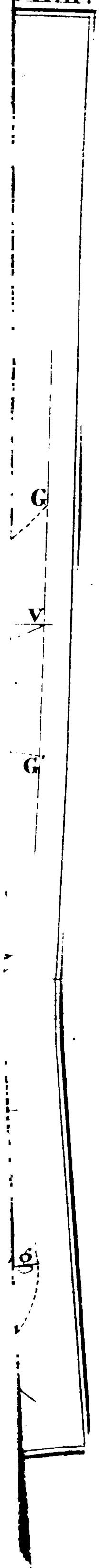
Second. Take, for the latter point, the intersection of the tangent ray, for each assumed point, with the vertical cylinder containing the given point of sight, and having a common axis with the torus.

Third. Find the intersection of the trial curve, joining these points, with the horizontal circle of the cylinder, through E,E′. A tangent, from this point to the torus, will be the revolved position of the visual ray, tangent at the required point. From it, the true position of the ray, and required point, can at once be found.

To complete the construction. Take, first, the point, B,B′, on the meridian parallel to the vertical plane. By the theory of curvature of surfaces (Des. Geom.), the normal sections (98) of maximum and minimum curvature at B,B′ are the meridian, whose radius of curvature is O′B′=R, and the section made by a plane, containing OB—O′B′, and perpendicular to the meridian plane pBE; for which section the radius of curvature is B′b=R′. Next, in the osculatory hyperboloid, at B,B′ (101), the gorge, and the meridian hyperbola, are the principal sections, respectively osculatory to the sections of the torus. Now let the axis on B′O′=$2c$, the other axis of the

* *From* LEROY, Stéréotomie, pp. 176–178.

XIII.



gorge $=2a$, and the definite conjugate axis of the hyperbola $=2b$. Then their lengths must be chosen so as to have

$$R = \frac{a^2}{c} \text{ and } R' = \frac{b^2}{c}$$

For simplicity, make $a=c$, as we are evidently free to do, for R may as well be the radius of the osculatory circle, as the equal radius of curvature of an osculatory ellipse at B,B′; since either curve may be the gorge of a hyperboloid. Then the osculatory hyperboloid becomes one of revolution, and

$$R=a, \text{ and } R' = \frac{b^2}{a}; \text{ whence } b = \sqrt{R \times R'} = B'D$$

a mean proportional, by construction, between $B'O' = a = R$, and $B'b = R'$.

This done, we see from (101, Second) that, for the tangents to the curve of contact of the visual cone to coincide with the element of this cone, at a point of that curve of contact, the conjugate diameters, to which they are parallel, must coincide. But as the conjugate diameters of any hyperbolic section of the osculatory hyperboloid approach each other, they finally coincide in the asymptote.

Therefore, in the problem before us, the visual ray (element of the tangent visual cone) which is tangent, at once, to the torus, and to its curve of contour, at B,B′, is parallel to the asymptote of any hyperbola cut from the osculatory hyperboloid by a plane, as AL; the meridian plane parallel to the tangent plane at B,B′ (101, Second).

Now this asymptote is a diagonal of the rectangle on the semi-axes $O'A = a$, and b; equal to B′D, but perpendicular to the circle of the gorge at O′, and the required parallel tangent at B,B′ is therefore the diagonal of a rectangle, whose sides are B′K′, equal and parallel to O′A, and a perpendicular to the vertical plane, at K′, and equal to $B'D = b$. Hence project K′ at K, by making $GK = B'D$, and BK—B′K′ is tangent, at B,B′, both to the torus and to the curve of contour due to a point of sight on BK—B′K′.

In like manner, as shown in full in the figure, find the similar tangents at other assumed points on the torus, as Q,Q′ and N,N′.

Coming now to the second operation (p. 176) find the points,

M,M′; R,R′; and S,S′, as shown in the figure, where the visual rays already found, pierce the vertical cylinder containing E,E′, and having the axis, p—Lp', of the torus, for its axis. These points are the several points of sight due to B,B′, etc., as the points where visual rays are doubly tangent, as already explained; and the curve, MS—M′N′S′, containing them, is the locus of all similar points of sight.

Hence, now, by the third operation (p. 176), E‴,E″, the intersection of this trial curve, MS—M′S′, with the horizontal circle through E,E′, and whose radius is pE, may be considered as the revolved position of E,E′ whence the visual ray E″T″—E‴T‴ is doubly tangent at T″T‴. Then, by counter-revolution, E″,E‴ returns, about the axis, p—Lp', to E,E′, and T″T‴, to T, through an angle of revolution, equal to E″pE. Likewise the second point, symmetrical with E″E‴, which evidently exists in front of the meridian plane RE, being revolved to E,E′, T″T‴ will then proceed, about p as a centre, in horizontal projection, to t,T. Then T,T′ and t,T′ are the points on the torus, where visual rays from E,E′ are tangent, both to the torus, and to its apparent contour.

Thus the like points, o,o'; o'',o'; v,v' and v'',v', Fig. 97, may be constructed, if desired.

6°—*To find the perspective of the apparent contour.*

This construction requires no operations not already explained in all previous problems where three planes have been used, as in Prob. XVI. Hence only a few points are shown in the figure. Taking any point of contour, as d,d', for example, the visual ray, dE—d'E′, from it, pierces PQP′, taken as the real position of the perspective plane, at x,x', which, when carried to the new position, pLp', of the perspective plane, and revolved to the left, as indicated by xx''—x'D″, parallel to the ground line, and $x''x'''$, with centre L, gives D′ the perspective of d,d'. In the same manner, B″, the perspective of b,b', and O‴, that of o,o', are found.

We thus see that the perspectives of o,o', etc., are cusps. It is often useful to find also the perspective of the circle, of radius OA, which separates the exterior and interior nappes of the torus.

DISCUSSION.

This discussion relates to the three general forms of the in-

tersection of a *vertical visual plane* with the interior half, or nappe, of an annular torus, and to the relations of their tangent visual rays to the whole torus.

FIRST. Any vertical visual plane, which intersects the gorge, will cut from the torus a curve of two branches of the general shape of Fig. 99.

The tangent planes to the gorge, as at u'' and u, intersect the torus in figure eight curves like Fig. 100.

Planes between Eu and Ev, contain curves like Figs. 101 and 102.

SECOND. The tangent visual rays, whose points of contact with these curves, are, geometrically speaking, points of apparent contour, are evidently seen to be also secants, intersecting the same curves, and therefore the torus, each in two points. It is further evident on inspection, that the point of contact, T, will either be in front of both points, a and b, of intersection, and therefore *visible;* or between them, as at t, Fig. 100, and t and t', Fig. 101; or beyond both, as at t, Fig. 99, t',t'', Figs. 100, 101. Also, that in both the latter cases, the point of contour t, or t', or t'', is *invisible*, that is, geometrically real, but perspectively unreal.

But when the ray, as in Fig. 102, is tangent to the contour, it evidently results from such a modulation of the curve of Fig. 101, that Tb and Atb' coincide, in a position like Tb, Fig. 102, so that, T, A, and t all coincide. Thus T is the common limit between both the visible part, containing A$'$, of the *curve of intersection* of the visual cone, with the torus, and the visible part of the apparent contour, on one hand; and the invisible part of the apparent contour on the other hand.

But the perspectives of the singular points, as T, Fig. 102, viz., of o,o'; o'',o', etc., Fig. 97, are the cusps O$'''$, etc., which are thus shown to be the limits between the visible and invisible portions of the apparent contour of the interior nappe of the torus.

THIRD. If, as shown by the rays at T$'$ and t', Fig. 99, the visual rays are so highly inclined that the eye can see through the gorge of the torus, the portion vhv''—$v'h'$ of the apparent contour will be visible; as will be completely evident on constructing the perspective under this condition.

FOURTH. The form of the tangent visual cone, and relations of the points o,o', etc., can be more fully apprehended by find-

ing the intersections of its elements with the horizontal plane, which will show its base to be a curved quadrangle, cusped at the four corners, as in Shades and Shadows (Prob. XXX.) where the base of the cylinder of rays tangent to the piedouche was a similar figure.

CHAPTER VII.

Perspectives of Reflections.

102. Since other than plane reflectors are commonly quite small and irregular, in comparison with ordinary mirrors, the images afforded by them are so obscure, distorted, or partial, that they can adequately be represented by purely imitative art, resulting from observation. Such, for example, are the reflections from polished knobs, or other decorations, from household articles of polished metal, or from polished cylinders of moderate size; none of which are generally worth the trouble of finding by construction.

103. Plane reflectors, of large size, occur both in nature and art, and it is therefore desirable to be able to construct the perspectives of the images formed by them. The following problems will afford ample illustrations of the perspectives of reflections from water surfaces taken as horizontal mirrors, and from vertical and oblique artificial mirrors.

104. The fundamental principle of reflection, is, that the incident and reflected rays make equal angles with the reflecting surface, or with a normal to it at the point of incidence, and are in the same plane with this normal.

See Pl. XIII., Fig. 91, where P is a luminous, or illuminated point, from which proceeds the ray PR, meeting the reflecting plane, QR, at R. There, the ray PR is reflected, so as to make the angle PRQ=ERK or PRN=ERN.

105. Now, *first*, the eye shapes itself to objects at different distances, so that within the greatest and least limits of distinct vision, the rays will converge in the eye so as to form the images of any objects within these limits. Hence, removing the mirror, the eye at E′ would justly image the point P. But, *second*, the rays, consecutive to PR, are reflected from R without change of relative position, and the eye refers objects to a position depending on the direction, and relative position, in which rays from them enter the eye; hence the eye at E, look-

ing in the direction ER would see the point P as if at P′, in the direction ER and at a distance EP′=E′P. Therefore P′, the image due to the reflection at R, is found by making PP′ perpendicular to QK, and P′Q=PQ.

106. As a first illustration, too elementary to compose a problem, take the case of the reflections of sunbeams, or other luminous rays, from a rippled surface of water, showing an extended line of reflections. All the incident and reflected rays, in this case, are in a vertical visual plane EPQ, Pl. XIII., Fig. 92, through the luminous centre S,S′. Hence, taking the water surface as the horizontal plane, ES, the horizontal trace of this plane, radiating from the observer, will be the line itself of reflected light. But, also, as this visual plane is vertical, the perspective of ES is PQ, the trace of that plane on the perspective plane, and PQ is perpendicular to the ground line. Therefore we have this principle, sometimes violated in painting. The *perspectives of reflections of light upon level water surfaces, are perpendicular to the ground line, that is, also, to the horizon.*

THEOREM XXIII.

When the reflecting surface is perpendicular to the perspective plane, the perspective of the trace upon it, of a plane, perpendicular to it and containing any number of given points, is an axis of symmetry between the perspectives of the given points and those of their reflections.

This theorem is true, simply because the normals, spoken of, being perpendicular to the reflecting surface, are therefore parallel to the perspective plane, when the reflecting surface is perpendicular to that plane.

Thus, in Pl. XIII., Fig. 91, let QK be the trace of a horizontal reflecting surface, on the plane of the paper; and let AB be the trace of the perspective plane, perpendicular both to the paper and the plane PQ. Then PP′, the line joining the point P with its image or reflection, P′, will be parallel to the perspective plane AB; and, being also bisected at Q, its perspective will not only be parallel to PP′, itself, but will be

bisected at the perspective of Q, which will be in the perspective of the trace of the plane PQRE upon the plane QK.

Problem XCIII.

To find the perspectives of the reflections, in planes perpendicular to the perspective plane, of lines taken at pleasure in planes perpendicular to the reflecting surface.

Here, Fig. 93, let EV be the horizon, E the vertical projection of the point of sight, and CC′ the vanishing line of a *vertical* plane, perpendicular to the perspective plane, and containing a rod, whose perspective is AB. The horizon, EV, is also the vanishing line of the reflecting water surface shown, and taken as the most familiar example of a reflecting surface, in reference to pictorial effects.

Now let A represent the intersection of the rod AB with the water, then AE will be the perspective of the trace upon the water surface, of the vertical plane containing AB. Also, C, represents the vanishing point of AB and of all parallels to it.

It now follows that lines, such as are represented by BB′, from a point B to its reflection B′, are parallel to the perspective plane, and they are bisected at their intersection with the water surface. Hence their perspectives, BB′, are bisected at the perspective of that intersection, that is, in the line AE. Hence make EC′=EC and AC′ will be the indefinite perspective reflection of AC. Also make MB′=MB, and AB′ will be the perspective of the reflection of AB.

The remainder of the figure illustrates the case where the vertical plane through the line is oblique to the vertical plane of projection. For, by the principles of vanishing lines (42), the vanishing line of any plane, perpendicular to both the horizontal, and the perspective planes, as in the last case, is a vertical line CEC′, perpendicular to the horizon, and through E, the centre of the picture. Therefore, as GG′, perpendicular to the horizon, does not pass through E, it is the vanishing line of a vertical plane which is oblique to the perspective plane. But otherwise, this case is similar to the preceding. Thus, if FV represents the perspective of the trace of this vertical plane upon the water surface; FK, a line in this plane; G, its

vanishing point, and F, its intersection with the water, then make $VG'=VG$, and $NK'=NK$, and F, being its own image, FK′ is the perspective of the reflection of FK in the water.

Once more, by simply turning the plate a quarter round, and by taking CC′ for the horizon, EV becomes the vanishing line of a vertical mirror, perpendicular to the perspective plane, as the side wall of the room, Fig. 94, and EA is the perspective of its trace on a horizontal plane containing the line AB. Then, still, the line represented by BB′, from a point itself, B, to its image B′, being perpendicular to the mirror, is parallel to the perspective plane, and is therefore bisected in perspective, as it is in reality by the mirror. Hence MB′, being equal to MB, as before, and A being the point of AB which is in the mirror, AB′ is the perspective of the reflection of AB in the mirror.

Problem XCIV.

To find the perspective of the reflection of the wall, steps, and arched inlet of a stone reservoir; in the surface of the enclosed water.

Let GL, Pl. XVI., Fig. 106, be the ground line, *on the surface of the water*, D′E′ the horizon, E′ the centre of the picture, D′ the real, and D, the reduced vanishing point of diagonals (Prob. LII.) taken to condense construction; by making $DE'=\frac{1}{2}D'E'$.

The principal dimensions are given in numbers, and used on a scale of half an inch to one foot, and all the numbers on the scale of depths GL (81) are numbered at half of the real value, of the depths themselves, since $E'D=\frac{1}{2}E'D'$. (Notice that depths here are not vertical depths of water, but depths back from the perspective plane.)

1°.—*To construct the front ends of the lower steps.* Let GZ, perpendicular to GL, be the zero line, for the scales of lengths and depths (81) it being the vertical trace of a plane of reference taken perpendicular to the ground line. Let the point C now be 3 feet to the right of GZ, and 7 feet back of the perspective plane, at B, then make $GB=3$ feet, by the scale, and $BC''=3\frac{1}{2}$ feet. Then BE′ and C″D are the perspective perpen-

dicular, and reduced diagonal (called simply the diagonal, for the sake of brevity), whose intersection is the initial point C, found in the perspective figure. Then CJ, parallel to GL, is the perspective trace of the foremost wall, on the water surface.

Let each step be 1 foot wide, and 9 inches high. Then make GJ=3 feet on the scale of heights, GZ, and GE′ and JE′ will be perspective perpendiculars (called for brevity the perpendiculars, simply), which limit the vertical height AA′ of the sum of the lower flight of steps.

Lines, parallel to the perspective plane and equally divided, are likewise divided in perspective, hence divide AC into three equal parts for the tread of the steps, and AA′ into four equal parts for their rise, and horizontal and vertical lines, as shown, through the points of division, will complete the front ends of the steps.

2°.—*To complete all the steps.* Let the steps be 3½ feet long, so that H′ for example is 10½ feet back of J. Then make JH=5¼ feet, and the diagonal HD will cut off the perpendicular A′E′ at H′, at the further end of the upper step. Next, draw perpendiculars, as CF, from all the front points, C, C′, *d′*, etc., of the steps, and limit them, as shown by vertical and horizontal lines, beginning at H′.

Suppose the landing, A′S′, to be 3 feet, by 7 feet. Then make A′N′=AC, and draw N′E′. The point S′ is 14 feet back of a point (J′) 3 feet to the left of J. We should therefore lay off 7 feet from (J′) that is JS=4 feet from J, and SD will then limit N′E′ at S′, from which draw S′P′ and a horizontal line, partly invisible, to complete the landing.

3°.—*To draw the upper steps.* Let the whole height of the arch wall above the water be 4½ feet, laid off on GZ; then ZE′ will limit H′X at the point X, through which the front edge, MT′, may be drawn. M is directly over *a″*, and M*a‴*=FF′, all the steps being alike.

The height of the wall back of the landing being 6 feet and 4 inches above the water, and the face containing Q being 1 foot to the right of GAA′, make B′E equal to that distance, and draw EE′. Make N′N″=1½ feet=½AA′, and draw N″E′, and from its intersection with S′P′, draw the horizontal joint to meet ME′; from the latter intersection draw the vertical to meet EE′ in Q, and complete the figure of the wall.

4°.—*To draw the arch.* Suppose FV to be 1 foot, then

make $BV''=1$ foot; and as the face of the arch is $10\frac{1}{2}$ feet behind the perspective plane, make $V''L=5\frac{1}{4}$ feet, and the perpendicular $V''E'$, and diagonal LD, will give V, a point in the left hand vertical edge of the arch way. Now suppose the highest point, l, of the arch to be 3 feet 9 inches above the water, make ss' at this distance above the ground line (and sV'' perpendicular to GL) and it will be the trace of the horizontal plane through l. Then make $ss'=5\frac{1}{4}$ feet, and the perpendicular sE', and diagonal $s'D$, will determine w, a point in the line VV' from which the horizontal line, wl, intersects YE', the centre line of the arch, at l, the perspective of the summit of the arch. Through l, draw the arc $U'V'$, of 60°, to limit the sides of the arch, and make the arch joints radiate from U. The point V''', in the rear edge of the arch, may be found by laying off from V'', on GL, its half depth back of the ground line, and drawing a diagonal to D.

THE REFLECTIONS.

5°.—*To construct the perspective reflections of the steps.* By (Theor. XXIII.) the image, or reflection, of each point is as far below the water surface, as the point itself is above the same surface.

Also, the reflection of every line parallel to the water is parallel to the line itself, and hence the two lines have the same vanishing point.

Hence, make $Cc=CC'$, and draw cf' towards E', and cf' will be the image of $C'F'$. Likewise, cd, equal and parallel to $C'd'$, is the reflection of that line; an', equal and parallel to $A'N'$, is the reflection of the latter line, and the like is true for all parts of the walls and steps; observing to make the axis of symmetry between a point and its image, the trace, upon the water, of the vertical face containing that point. Thus mt is as far below FW, as MT is above it.

The image P, of P', is found, conveniently, by making $NO'=NO$ and drawing $O'E'$, on which P' is reflected, at P, in the line PP', perpendicular to the ground line. Then, through P, the parallel reflection of $P'Q$ is drawn, which is but partly visible.

6°.—*The reflections of the arch and rod.* The centres, U and u', are equidistant from VW, and the reflections, as tu, of

the radial joints all tend to u'. The joints, whose reflections are ug, qb, etc., are not seen, but, being perpendiculars to the perspective plane, their visible reflections all tend to E′. The reflections, ug and vn, limit the rear vertical edges at V‴ and W′, and the water might be shown at pleasure beyond the horizontal line V‴W′, which is the line of symmetry between ng, and its unseen perspective original.

Let the insertion of the rod, RK, into the wall, be at the intersection of AA′ and C′d', and let its free extremity, R, be located at pleasure on the horizontal line through a'. (The student should construct R, from assumed distances to the left of A, that is of G, and above the water, that is above GL.) Suppose R to be 5 feet back of the perspective plane, then make BB″, for example = $2\frac{1}{2}$ ft., and draw B″D to intersect BE′ at y', through which y'R′ is drawn parallel to GL. Then R′, where y'R′ intersects the vertical Rr, is the perspective intersection of that vertical with the water; hence make R′r= R′R to find r, the reflection of R. Also AR′ would be the axis of symmetry between points of RK, and their reflections, since Ak is evidently equal to AK, giving kr for the required reflection of KR.

EXAMPLE 1.—*Find the perspective of the reflection of a room, and of objects within it, in a mirror which shall coincide with a side wall, perpendicular to the horizontal and perspective planes.*

Ex. 2.—*Having the perspective of a tank, having a square opening, in the top of a table, and of prisms, cones, etc., standing near the further edge of the tank, find the perspective of their reflections in the tank, the latter being full of water.*

THEOREM XXIV.

When a reflecting surface is oblique to the perspective plane, the intersection of the mirror, with a plane perpendicular to it, through any line, will no longer be an axis of symmetry.

This theorem is true, simply because the normals to the mirror from points to their reflections will not be parallel to the perspective plane; hence they must have a vanishing point.

Also, the perspectives of equal distances on such lines will be unequal; hence the intersection, or trace, described cannot be, as before (Theor. XXIII.), the bisecting line of the perspectives of these normals.

This theorem is most obviously true, when, for example, the mirror is parallel to the perspective plane, for, then, the normals, described, being perpendiculars, their perspectives will vanish at the centre of the picture; as in the following problem.

PROBLEM XCV.

To find the perspective of the interior of a room, and of a vertical staff therein and of their images; the rear wall of the room being a mirror, parallel to the perspective plane.

The construction of the perspective of the room itself, Pl. XIII., Fig. 94, which is represented as square, requires no special explanation, as it requires no new operations. The staff, *gk*, located by the method of scales (81 ....) is at the distance AG from the left wall, the distance AH from the perspective plane, AB being the ground line, and is of the height shown at AK.

The lateral edges of the floor and ceiling are perpendicular, both to the mirror, *abcf*, and to the perspective plane. Hence, as in Fig. 91, the image P′Q, of PQ, is perpendicular to the mirror QK, so here, the image of the original of A*a* is A*a* produced; that is, it is a perpendicular. Hence its perspective is A*a* produced, or *aa*′, limited by the reduced diagonal BD (not shown) where E′D=$\frac{1}{2}$E′D′, since AB=$\frac{1}{2}$ of double the depth of the room. Then *a*′*b*′*c*′*f*′ is the perspective of the image of the section ABCF of the room, made by the perspective plane.

Next to find the perspective of the image of the staff, *gk*, in the most elementary manner, make AM=AB, and MN will be the plan of the rear wall of the room. Then through G, the distance of *gk* from the left wall, draw GP parallel to AM, and make *m*P=HB; and P will be the horizontal projection of the image of the staff. Now GE′ is the indefinite perspective of GP, and the vertical visual plane PE, drawn by making IE= E′D′ on E′I, a perpendicular to the ground line, AB, meets AB

at L, whence its vertical trace Lg' meets GE′ in g', the perspective of the image of g.

Otherwise: make AO′=GP, then, by the method of reduced distances AO=¼AO′, and OD will intersect AE′ at n, the perspective of the image of h, and ng', parallel to AB, being the perspective of the image of hg, will meet GE′ at g', the perspective of the image of g, as before.

Then k', the perspective of the image of k, is the intersection of $g'k'$ with kE′, giving $g'k'$ for the perspective of the image of gk; remembering that the line in space from k to k' is, like PP′ in Fig. 91, perpendicular to the mirror, and therefore, in this case, to the perspective plane, also.

Problem XCVI.

To find the perspective of the reflection of any object, in a mirror whose plane is oblique to both planes of projection.

This problem, Pl. XIII., Fig. 95, answers to the case of a mirror in a corner of a room, and inclined to the floor and to both the adjacent walls.

1°.—*The perspective of the given object and of the mirror.*

Let PQP′ be the plane of the mirror, given by its traces, and let the given object be the square, ABCF—hH′, contained in the horizontal plane whose vertical trace is G′Q′, at the height Q′q above the horizontal plane of projection.

The position of ABCF indicates the revolution of the horizontal plane through 180°, which brings E, the horizontal projection of the point of sight (station point (17)) at a distance, Ed, behind the ground line VQ, equal to E′D, where D is taken as the vanishing point of diagonals. PQ is shown however as if not revolved.

Then make the angle fEV=PQg, and Ef, parallel to the ground line, and EV—E′V′ will represent the visual ray, parallel, in its primitive position, to PQ, and therefore piercing the perspective plane at V′, the vanishing point of PQ, and of the parallel trace of PQP′, on *any* horizontal plane, as G′Q′.

The trace of PQP′ on the latter plane, evidently pierces the perspective plane at Q′, hence Q′V′ is the perspective of that trace; giving P′Q′V′ for the perspective of the mirror.

The perspective, *abcr*, of the given square is found by diagonals and perpendiculars.

2°.—*The perspective of one point of the reflection.*

E′Y, perpendicular to P′Q, is the vertical trace of a visual plane perpendicular to the mirror, and to the perspective (or vertical) plane. This plane will therefore contain visual rays both parallel and perpendicular to the mirror. But V′*e*, parallel to P′Q (Theor. IV.), is the vanishing line of the plane of the mirror; hence, making E′E″=E*d*, and perpendicular to *e*E′Y, we have E″*e* as the revolved position of a visual ray parallel to the mirror, and E″Y, perpendicular to E″*e*, as the revolved position of a visual ray normal to the mirror. The latter ray pierces the perspective plane at Y, which is therefore the vanishing point of all normals to the mirror; hence *a*Y, *c*Y, etc., are the perspectives of such normals from the corners of the square. Now *hk*, perpendicular to P′Q, is the vertical projection of such a normal, and Q″K, parallel to E″*e*, is the revolved position of the intersection of the mirror with the vertically projecting plane of this normal, hence *h*K is the revolved position of the normal. In counter revolution, K returns, perpendicularly to *hk*, to K′, the vertical projection of the intersection of the normal with the mirror. Then make K′*k*=K′*h*, and *k* will be the vertical projection of the reflection of the point A,*h* (whose perspective is *a*). Then *a*′ the intersection of the perspective, *k*E′, of the perpendicular through *k* with *a*Y, the perspective of *hk*, is the perspective of the reflection of *a*.

3°.—*To find the remaining points of reflection.*

The intersections, *m*, *n*, etc., of the indefinite sides of the square, with V′Q′, the perspective of the trace of the mirror upon the plane of the square, are points in the indefinite images of those sides. Hence *b*′, the reflection of *b*, is the intersection of *ma*′ with *b*Y, the perspective of the normal from *b*. In like manner, *c*′ is the intersection of *a*′*n* with *c*Y.

Each point might however have been found independently, as was *a*′.

By drawing the revolved position of QP, through Q, and parallel to EV (1°.) it would cut off a corner of ACFB, as it should to correspond with the same intersection, in perspective, at *or*.

After fully comprehending the last two problems, there will be no difficulty in solving the following examples, which are intermediate in complexity between the two just given.

EXAMPLE 1.—*Find the perspective of a trapezoidal room* 12 *feet long by* 10 *feet deep, whose vertical side walls make an angle of* 70° *with the rear wall, and the perspective plane, and let one of these walls be a mirror and let the other contain a window, door, fire-place, or picture; and let the reflections of the whole be constructed.*

Ex. 2.—*Let the mirror, still vertical, make an angle of about* 20° *with the rear wall.*

Ex. 3.—*Let the mirror be perpendicular to the perspective plane, but inclined* 70° *or* 80° *to the floor, the room, now, being square cornered.*

CHAPTER VIII.

Perspectives of Shadows by Candle-light.

107. A single comprehensive example, of the most usual kind, viz. shadows in domestic apartments lighted by one light, will suffice to illustrate this branch of perspective construction.

Problem XCVII.

To find the various shadows in the interior of a room, cast by objects illuminated by a candle.

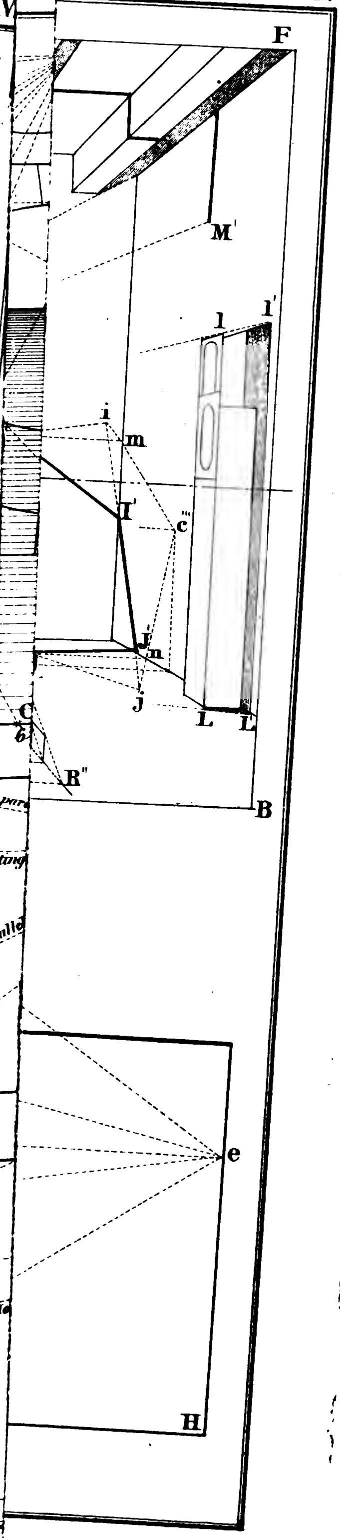
Pl. XVI., Fig. 107. This figure is taken, reduced in size, and with some alterations, from the elegant collection of perspective examples by J. Adhemar.

1°.—*Preliminary.* The perspective of the room itself, and its furnishings, is supposed to be given, it being found by the method of scales (81....) or by any of the methods already frequently illustrated. We therefore proceed at once with the construction of the several shadows; only premising, that constant use is made of the projections of the flame, C, upon the walls, floor, and ceiling of the room. These projections, that is, strictly speaking, their perspectives, are c, upon the floor; c', upon the rear wall; c'', upon the left hand wall; c''', upon the right hand wall; C', upon the plane of the undersides of the rafters; and C'', upon the ceiling.

Also, E is the centre of the picture.

2°.—*The shadow of the table.* The table is, for simplicity, a skeleton table, whose top is an opaque rectangle. Ck is the perspective ray which meets $c'k$, its perspective horizontal projection, at k, the perspective shadow of the front right hand corner of the table. The edges of the table, being parallel and perpendicular to the rear wall, ka, parallel to the ground line AB, is the shadow of a part of the front edge, upon the floor. Then ac'' is the indefinite shadow of the same edge upon the left hand wall, and it is limited by the ray Cb', through the left

Pl. XVI.



hand front corner of the table, or by bb', the trace, on this wall, of the vertical plane of rays, $Ccbb'$. Thus either of the points, c'', b', or a, is a check upon the other two.

From b', the shadow of the left edge of the table vanishes at E, and is limited at e', which is found just as b' was. Then $c''e'$, produced, gives the shadow of a part of the back edge of the table upon the left wall; and at f, where this shadow meets the rear vertical edge of the room, the horizontal shadow, fg, on the rear wall begins.

ch is the trace, on the floor, of another vertical plane of rays through C; hh' is its trace on the door, and h', in the ray Ch', is the shadow of the right hand back corner of the table, on the door. Then join h' with g. Now hK, the trace of the door, on the floor, meets kE, the indefinite shadow of the right edge of the table, at K; hence draw Kh', for the indefinite shadow of the right hand edge of the table on the door, and this will complete the shadows of the table.

3°.—*The shadows of the shelves.* These are simply found by combining the rays from C, with their projections from c' and c''. Thus, CQ meets its projection, c''P, at q, the shadow of Q on the side wall. From C, draw CY to meet its projection $c'y$, and y will be found. yr is parallel to the horizon, and rs tends towards c'', it being, like $c''a$, etc., the trace, on the left wall, of a plane of rays perpendicular to that wall, since the front edge, from Y, through which this plane is drawn, is perpendicular to the side wall.

Finally, qs is drawn to E, since the edges of the side shelf, beginning at P and Q, vanish at E, also.

4°.—*The shadow of the door, and adjacent cricket.* The trace, cH, of the plane of rays through GH, meets the base of the cricket at u, and the back edge of the floor at U. Then the vertical line at U, limited by the ray CG, is the shadow of GH on the rear wall, and $U'G'$ is the shadow of GG'. From u, there proceeds a vertical shadow, which turns at the upper edge of the cricket, and proceeds to meet the line UU' in the back edge through T'.

Produce $c'T'$ to t', and $t't$, drawn from E, and limited by CT, is the shadow of TT' on the floor. Or, $t't$ may be limited by the line from c through the foot of the vertical edge at T, which bounds the shadow of this edge on the floor. $T't'$ is the shadow of a part of TT' on the rear wall.

5°.—*The shadows of the rafters and hanger.* The small shadows of the rafters, on the rear wall, are the traces, on that wall, of planes of rays perpendicular to it, and containing the lower edges of the rafters.

Their shadows on the ceiling, are the traces of the same planes on the ceiling.

The former traces, as $c'Vv$, are drawn from c'; the latter, from the points, as v, tending towards E.

The *left hand* lower edges of the two left hand rafters cast invisible shadows.

The shadow of the hanger, MN, begins at N, and radiates from C″, the projection of C upon the ceiling, to N′. It is then vertical, and then radiates from C′, the projection of C on the plane of the undersides of the rafters, and continues to be found in the same way, till ended, at M′, by the ray CM.

6°.—*The shadows of the staff and clock.* IJ is a staff. Jn, parallel to AB, is the trace, on the floor, of a plane of rays, through J, and perpendicular to the side wall, and $c'''n$ is its trace on the side wall; which, being limited at j, by the ray CJ, gives j as the shadow of J on the wall produced.

Likewise: Im, parallel to AB, is the trace on the rear wall, of a plane of rays perpendicular to the side wall, and $c'''m$ is its trace on the latter wall. Then i, where $c'''m$ meets the ray CI, is the shadow of I on the side wall produced; and ij is the indefinite shadow of IJ on the same wall. Hence II′, and JJ′, are the shadows on the rear wall and floor.

The shadow of the clock is very simply found, by the trace cL, produced to L′; the vertical shadow and trace L′l', limited by the ray Cl; and the line from l towards E, whose limit is obvious.

In the preceding shadow, j might have been found by a vertical plane of rays, as b' and l' were.

7°.—*The shadow of the cricket on the hearth.* Suppose a fire-place to be in the front wall of the room, which is not in the picture; and let the cricket R′ be near it.

Any point of shadow, as r', is at the intersection of the ray Cr', with its horizontal projection cR′; but some of the intersections so found are too acute, hence the following auxiliary operation is employed. Any line, as ER″, is the horizontal trace of an auxiliary vertical plane, perpendicular to the perspective plane, and the quadrilateral at S is the perspective of

the projection of the cricket upon it. Also c'''' is the projection of C upon the same plane. Then $c''''R''$, through a corner of S, is the projection of CR and Cr'. Hence, draw $R''R$, parallel to AB, to meet CR and Cr' at the points of shadow, R and r'. The point, back of R, is similarly found, as is shown in the figure.

108. In more complicated problems than the last, where, for example, shadows made by a candle, upon curved surfaces might appear, it would often be easier to find their projections first.

Shadows on cylinders may however be readily found, when the latter are vertical, or have their axes *horizontal*, and either parallel or perpendicular to the perspective plane.

EXAMPLE 1.—*In the last problem let the ceiling be cylindrical, and with its axis perpendicular to the perspective plane.*

Ex. 2.—*Let the axis of a cylindrical ceiling be parallel to the ground line.*

Ex. 3.—*Let the room be a vertical cylinder, surmounted by the frustum of a cone of equal base with the cylinder, as in the attic room of a circular French roofed tower.*

(In this example find where the ray from the candle to the vertex pierces the cone's base.)

CHAPTER IX.

Distorted Perspectives: Singular and Anamorphous.

109. Distorted perspectives may be of several kinds; such as those which are due to an unnatural obliquity to the perspective plane, of the visual rays proceeding from the object; and those called *anamorphoses.*

An anamorphosis, is an ascertained figure, whose reflection in a curved mirror shall coincide with the perspective of a given figure.

Thus, let there be a vertical cylindrical mirror, and a circle or other figure, in the perspective plane, this given figure being the perspective of some given figure in space. It is then required to find a figure in the horizontal plane (or some other surface, in the most general case) such, that those rays from it, which are reflected from the mirror to the eye, shall coincide with those coming directly from the known figure in the perspective plane.

Anamorphoses, being merely perspective curiosities, or even trifles, no examples of them are here given.

110. The former kind of distorted perspectives always appear, more or less clearly as such, when a perspective figure, drawn from an unnaturally placed point of sight, is viewed from ordinary points of view. But when, for example, the eye is *very far* from the base of the picture, or ground line, in proportion to its distance from the perspective plane, the distorted images, may be called *singular perspectives.*

These are sometimes of some utility, and, accordingly, this volume closes with a single example of them.

PROBLEM XCVIII.

Construction of a singular perspective.

Let $gLkh$, Pl. XVII., Fig. 114, be the plane, revolved 180° about GL, of some ground, really in front of the perspective

plane, and divided into auxiliary squares, with a creek and a house; and let GL be made to coincide with G′L′, the ground line, and base of the picture; and let *kh*GL be perpendicular to the plane of the paper. Then let E,E′ be the point of sight, at the distance E′N′ below the horizontal plane, and at the distance E′D′=EN′ in front of the perspective plane, as illustrated in the small auxiliary end view, Fig. 116, which is similarly lettered, and in which *e* is the point of sight itself, in space. D′, being thus the vanishing point of diagonals, D′*g*′H is the perspective of *gh*. E′ being the vanishing point of perpendiculars, E′H, etc., are the perspectives of L*h*, etc., and these meet *g*′H at M, etc., the perspectives of *m*, etc., in the parallels to GL. Having thus found the perspectives of the auxiliary squares, the perspective of the creek, etc., can be sketched by hand; and the use of the perspective is to show an enlarged view, as seen from ordinary points of vision, of such details as it may be desirable to show on a larger scale than the rest of the picture.

In this example, E and E′ coincide. The full construction of D′, considered as the vanishing point of any horizontal line, as *gh*, is, to draw ED, parallel to the real position of *gh*, and then to project D into E′D′, a parallel to GL, at D′.

111. A *third species* of what habit leads one to call a distorted perspective, is produced when the perspective plane is inclined to the vertical edges of an object many of whose principal lines lie in a vertical direction.

Such lines, being no longer parallel to the vertical or perspective plane, would converge, in perspective, to a vanishing point. Yet the *impression* of a vertical position to lines *really* vertical, is so strong in observing the real object from any point of view, that the perspective which represents them otherwise seems strangely distorted. But observe that the distortion will disappear on viewing the perspective figure exactly from the actual point of sight.

The case just described is illustrated in photographs of buildings above or below the operator, when the camera has to be inclined up, or down, so that the paper in it is not vertical.

THE END.